Vectorial elements for the Galactic disc tide effects in cometary motion

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ABSTRACT

The classical Matese–Whitman theory of Oort Cloud comet perturbations has been revisited and extended. An explicit solution for the motion of the mean ascending node is given; it involves an elliptic integral of the third kind. Equations of the mean orbit are formulated in terms of the Cartesian components of the Laplace and angular momentum vectors (vectorial elements). The equations are solved in terms of elliptic functions and the solution is free of the ambiguity related to the orientation of the perihelion that was present in previous work. The Cartesian equations of motion for the vectorial elements form a Hamiltonian system of the Lie–Poisson type. This allows them to be integrated numerically by means of Hamiltonian splitting methods. The formulae of such an integrator are derived with a Hamiltonian function split into two parts.

Key words: methods: analytical – methods: numerical – celestial mechanics – comets: general – Oort Cloud.

1 INTRODUCTION

The Oort Cloud (Oort 1950) is commonly considered to be the main source of the comets observed in the Solar system. The gravitational effect of the Galaxy plays a fundamental role in the evolution of the Oort Cloud comets because - unlike stellar or molecular cloud encounters - it is the only significant force that acts systematically. Since the first work of Chebotarev (1966) and Byl (1983), the Galactic force has been treated in the tidal approximation. Heisler & Tremaine (1986) influenced further studies by indicating the dominant role of the Galactic disc and using an averaged Hamiltonian function of this problem. Matese and Whitman observed that the averaged Hamiltonian of the Galactic disc tides leads to equations of motion directly solvable in terms of elliptic functions and integrals (Matese & Whitman 1989, 1992). Most of the present works extensively exploit their solution, but at the same time we can observe that more attention is being paid to the action of the Galactic Centre (Brasser 2001; Fouchard 2004). The latter effect, albeit small, has a particular importance: it destroys the axial symmetry of the problem. Without the axial symmetry, the Galactic potential averaged with respect to the mean anomaly still has two degrees of freedom and is apparently non-integrable. This explains why the averaged system with the Galactic Centre tide included has always been numerically integrated.

The present paper is the first step towards an analytical treatment of the complete tidal potential of the Galaxy. In principle, it is entirely dedicated to the Galactic disc effects: we aim at improving and extending the classical solution of Matese & Whitman (1989). First, we fill in two gaps in the Matese–Whitman theory: we present an explicit formula for the evolution of the orbital ascending node; and we resolve the north–south ambiguity in the location of the perihelion. Both deficiencies were to some extent tolerable in the Galactic disc framework, but they have blocked the way towards a general Galactic tide model. The second improvement is significant even for the Galactic disc effects alone: recalling the significance of the Laplace vector discussed in Breiter, Dybczyński & Elipe (1996), we formulate the averaged Galactic disc perturbations problem in terms of the Laplace vector and angular momentum vector components. The resulting equations are elegant, simple and non-singular. They can be numerically integrated using a Lie–Poisson splitting method.

2 KEPLERIAN ELEMENTS SOLUTION

2.1 Preliminaries

Let us introduce a heliocentric reference frame with the Oxy plane parallel to the Galactic disc. In this reference frame a comet in the Oort Cloud attracted by the Sun and by the Galactic disc obeys equations of motion

$$\ddot{r} = -\frac{\mu}{r^3}r - \frac{\partial V_{\rm d}}{\partial r},\tag{1}$$

where $\mu = GM_{\odot}$ is the heliocentric gravity parameter and the Galactic disc potential is

$$V_{\rm d} = 2\pi G \rho z^2,\tag{2}$$

where ρ is the local density of the Galaxy and *G* stands for the gravitational constant. For the convenience of further considerations, we

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introduce a dimensionless parameter ε defined as

$$\varepsilon = 2 \, \frac{\sqrt{\pi G \rho}}{n_0},\tag{3}$$

where $n_0 = \sqrt{\mu a^{-3}}$ is the Keplerian orbital mean motion of a comet. Averaging the perturbing potential V_d with respect to the mean anomaly we obtain

$$\langle V_{\rm d} \rangle = \mathcal{K}_1 = \frac{1}{4} \varepsilon^2 n_0^2 a^2 s^2 (1 - e^2 + 5e^2 \sin^2 \omega).$$
 (4)

The symbols that appear in equation (4), namely the semimajor axis a, sine of inclination $s = \sin I$ (analogously, we will also use $c = \cos I$), eccentricity e, argument of perihelion ω , and the mean motion n_0 , have the meaning of mean orbital elements; they differ from the osculating elements by the absence of short-period perturbations. Both a and n_0 are constant as the effect of removing the mean anomaly from \mathcal{K}_1 .

Resorting to the Hamiltonian formulation one immediately obtains the prime integral

$$\sqrt{(1-e^2)}\,c = \alpha = \text{constant},\tag{5}$$

which is a consequence of the invariance of \mathcal{K}_1 with respect to rotations around the Oz axis. Thus, in order to determine the shape and orientation of a mean elliptic orbit, only three Keplerian elements are required: ω , Ω and either *I* or *e*, provided the constants *a* and α have been specified. In further discussion the knowledge of *a* will always be implicitly understood. A second integral of motion is obviously $\mathcal{K}_1 = \text{constant}$; combining it with equation (5) and rejecting all terms known to be constant, we obtain the reduced energy integral

$$e^{2}(1-5s^{2}\sin^{2}\omega) = \beta = \text{constant},$$
(6)

introduced by Breiter et al. (1996). The sign of β plays an essential role, helping to distinguish solutions with librating ($\beta < 0$) or circulating ($\beta > 0$) argument of perihelion. Note also that β is the minimum value attained by the square of eccentricity in the circulating ω regime – a fact that can be immediately checked by setting $\omega = 0$ or $\omega = \pi$ in equation (6). If we specify both α and β , then ω and Ω uniquely define a mean orbit. In order to simplify the final results, we introduce four auxiliary parameters:

$$\gamma = \frac{1}{4}(4 - 5\alpha^2 - \beta),$$
(7)

$$\kappa = \sqrt{\gamma^2 + \beta},\tag{8}$$

$$\xi_1 = \frac{1}{2}(\gamma + \kappa),$$

$$\xi_2 = \frac{1}{2}(\gamma - \kappa). \tag{10}$$

The parameters are closely related to the quantities that appeared in Matese & Whitman (1992):

$$H_{z}L^{-1} = \alpha,$$

$$H_{0}^{2}L^{-2} = 1 - \beta,$$

$$H_{1}^{2}L^{-2} = 1 - \xi_{1},$$

$$H_{2}^{2}L^{-2} = 1 - \xi_{2}.$$
(11)

Similar symbols were used in Breiter et al. (1996), but some of them have a different meaning in the present paper. Using the subscript 96 for the quantities from Breiter et al. (1996), we have

$$\alpha^2 = \alpha_{96}, \qquad \beta = \beta_{96}, \qquad \kappa^2 = \frac{1}{16}\kappa_{96},$$
 (12)

as well as $\varepsilon_{96} = \varepsilon n_0$. We recall that ξ_1 is the maximum eccentricity squared value that can be reached for given α and β .

For convenience, we introduce a dimensionless time variable τ , such that $\tau = 0$ when the evolving eccentricity attains one of its maxima, and

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \varepsilon^2 n_0. \tag{13}$$

Derivatives with respect to τ will be marked by a 'prime', e.g. $t' = \varepsilon^{-2} n_0^{-1}$, and derivatives with respect to time by a 'dot', i.e. $\dot{\tau} = \varepsilon^2 n_0$.

2.2 Eccentricity

The classical solution of Matese & Whitman (1989, 1992) provides an explicit formula for $\eta = \sqrt{1 - e^2}$, and hence for the eccentricity *e*, allowing one to determine inclination from equation (5) and 2ω from the energy integral. After minor transformations the solution of Matese & Whitman can be cast into the form¹

$$e^{2} = \xi_{1} - A^{2} \operatorname{sn}^{2}(B\tau \mid m), \qquad (14)$$

where the elliptic modulus is

$$m \equiv k^2 = \frac{A^2}{B^2},\tag{15}$$

with

(9)

$$A = \sqrt{\xi_1 - \max(\beta, \xi_2)},\tag{16}$$

$$B = \sqrt{\xi_1 - \min(\beta, \xi_2)}.$$
(17)

Actually, the solution type is determined by the sign of β , because $\beta > 0$ implies $\beta > \xi_2$, while $\beta < 0$ means $\beta < \xi_2$ (compare equations 8 and 10). When $\beta = 0$ we obtain the homoclinic motion with

$$e^{2} = \frac{1 - \frac{5}{4}\alpha^{2}}{\cosh^{2}\left[\sqrt{\left(1 - \frac{5}{4}\alpha^{2}\right)\tau}\right]},$$
(18)

which asymptotically tends to $e^2 = 0$. The inequality $(1 - 5/4\alpha^2) \ge 0$ results from the definition of $\gamma = \xi_1 + \xi_2$, combined with $\beta = 0$ and non-negative values of ξ_1 and ξ_2 (see Matese & Whitman 1989, 1992).

2.3 Ascending node longitude

The ascending node longitude Ω is a variable seldom mentioned in the discussion of the Galactic disc perturbations. Owing to the axial symmetry of the problem, the Lagrange equation (Brouwer & Clemence 1961)

$$\Omega' = -\frac{c}{\varepsilon^2 n_0^2 a^2 s \sqrt{1 - e^2}} \frac{\partial \mathcal{K}_1}{\partial s}$$
$$= -\frac{c(1 - e^2 + 5e^2 \sin^2 \omega)}{2\sqrt{1 - e^2}}$$
(19)

can be detached from the remaining system and solved by quadratures. Substituting prime integrals (5) and (6), we can simplify equation (19), obtaining a form that depends only on constants and e^2 :

$$\Omega' = -\frac{\alpha(1-\alpha^2-\beta)}{2(1-\alpha^2-e^2)}.$$
(20)

¹ In this paper we use elliptic functions sn, cn and dn.

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Maintaining the assumption that $\tau = 0$ at the instant when e^2 attains one of its maxima, and assuming that the ascending node of a mean orbit at that epoch has longitude Ω_0 , we solve equation (20) by quadratures as

$$\Omega = \Omega_0 - \frac{1}{2}\alpha(1 - \alpha^2 - \beta) \int_0^\tau \frac{d\tau}{1 - \alpha^2 - e^2(\tau)}.$$
 (21)

Substituting the expression for e^2 from equation (14), we obtain

$$\Omega = \Omega_0 - \frac{\alpha}{2} \frac{1 - \alpha^2 - \beta}{1 - \alpha^2 - \xi_1} \int_0^\tau \frac{\mathrm{d}\tau}{1 + \left[A^2 / \left(1 - \alpha^2 - \xi_1\right)\right] \operatorname{sn}^2(B\tau \mid m)}.$$
(22)

Introducing the Jacobi amplitude

$$\varphi = \operatorname{am}(B\tau \mid m), \tag{23}$$

with

$$B \,\mathrm{d}\tau = \frac{\mathrm{d}\varphi}{\sqrt{1 - m\sin^2\varphi}},\tag{24}$$

we can change equation (22) into

$$\Omega = \Omega_0 - C \int_0^{\varphi} \frac{\mathrm{d}\vartheta}{(1 - n\sin^2\vartheta)\sqrt{1 - m\sin^2\vartheta}},\tag{25}$$

where

$$C = \frac{\alpha}{2B} \frac{1 - \alpha^2 - \beta}{1 - \alpha^2 - \xi_1}$$
(26)

and

$$n = -\frac{A^2}{1 - \alpha^2 - \xi_1} \leqslant 0.$$
⁽²⁷⁾

The quadrature in equation (25) is nothing other than the definition of the incomplete elliptic integral of the third kind with argument φ , modulus *m* and parameter *n* (Byrd & Friedman 1954; Abramowitz & Stegun 1972). Hence

$$\Omega(\tau) = \Omega_0 - C \Pi(n; \varphi(\tau) \mid m).$$
⁽²⁸⁾

A negative value of *n* implies the so-called circular case of the elliptic integral of the third kind.

The evolution of Ω takes the form of periodic oscillations $\Psi(\tau)$ superimposed on a linear drift $-C_0 \tau$, namely

$$\Omega = \Omega_0 - C_0 \tau - \Psi(\tau). \tag{29}$$

The drift rate C_0 can be deduced from the property

$$\mathbf{\Pi}(n; \varphi + k\pi \mid m) = 2k \mathbf{\Pi}(n \mid m) + \mathbf{\Pi}(n; \varphi \mid m), \text{ for } k \in \mathbb{Z},$$

where

$$\mathbf{\Pi}(n \mid m) \equiv \mathbf{\Pi}(n; \, \pi/2 \mid m),\tag{31}$$

is the complete elliptic integral of the third kind. So, knowing that elliptic amplitude φ increases by $\pi/2$ in the $B^{-1} \mathbf{K}(m)$ interval of τ , where \mathbf{K} stands for the complete elliptic integral of the first kind, we conclude that

$$C_0 = \frac{CB\,\Pi(n\mid m)}{K(m)}.\tag{32}$$

The periodic part $\Psi(\tau) = \Psi(\tau + T)$ has a period

$$T = 2B^{-1} K(m), (33)$$

the same value as the period of eccentricity oscillations.

2.4 The ω problem

Three of the four mean Keplerian elements can be directly computed by means of the expressions we have provided. Eccentricity $e(\tau)$ is simply the square root of $e^2(\tau)$ given by equation (14) or (18). The longitude of the ascending node $\Omega(\tau)$ is explicitly defined in equation (28). The inclination $I(\tau)$ can be deduced from $e(\tau)$ and α

$$I(\tau) = \arccos\left[\frac{\alpha}{\sqrt{1 - e^2(\tau)}}\right].$$
(34)

What about the argument of perihelion? Matese & Whitman (1989, 1992) simply propose to use the energy integral, for example computing

$$(\sin \omega)^2 = \frac{(e^2 - \beta)(1 - e^2)}{5(1 - \alpha^2 - e^2)e^2}.$$
(35)

In these circumstances we are unable to distinguish values of ω that differ by π . This ambiguity is by no means accidental: it reflects the fundamental symmetry properties of the Galactic disc potential (symmetry axis perpendicular to the plane of symmetry z = 0). For the same reason it has not been considered a serious drawback as far as the disc perturbations alone are considered. However, the situation will change if one is interested in the disc perturbations as the main, unperturbed part of a more general problem like the complete Galactic potential influence on cometary orbits. This is our motivation to look for an unambiguous solution that allows a proper identification of ω within the [0, 2π) interval. Although this problem can be solved within the Lagrange equations framework, we suspend it until Section 3, where it will be solved more easily thanks to the use of vectorial elements.

3 VECTORIAL ELEMENTS

3.1 Equations of motion

(30)

In a problem defined by a potential averaged with respect to the mean anomaly, elegant and symmetric equations exist for the time derivatives of the Laplace vector

$$\boldsymbol{e} \equiv \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = e \begin{pmatrix} \cos\omega \cos\Omega - c\sin\omega \sin\Omega \\ \cos\omega \sin\Omega + c\sin\omega \cos\Omega \\ s\sin\omega \end{pmatrix}, \quad (36)$$

and of a dimensionless angular momentum vector

$$\boldsymbol{h} \equiv \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \sqrt{1 - e^2} \begin{pmatrix} s \sin \Omega \\ -s \cos \Omega \\ c \end{pmatrix}.$$
(37)

These 'vectorial elements' were initially introduced instead of Keplerian elements by Milankovitch (1939). They reappeared in papers by Musen (1961) and by Allan & Ward (1963). Allan & Cook (1964) provided a simplified form of Musen's equations, valid for an averaged system without explicit time dependence,

$$\dot{\boldsymbol{h}} = \boldsymbol{h} \times \frac{\partial V^*}{\partial \boldsymbol{h}} + \boldsymbol{e} \times \frac{\partial V^*}{\partial \boldsymbol{e}},\tag{38}$$

$$\dot{\boldsymbol{e}} = \boldsymbol{e} \times \frac{\partial V^*}{\partial \boldsymbol{h}} + \boldsymbol{h} \times \frac{\partial V^*}{\partial \boldsymbol{e}},\tag{39}$$

where V^* was an averaged perturbing potential $\langle V \rangle$ divided by $n_0 a^2$. The authors noticed two identities,

$$\boldsymbol{h} \cdot \boldsymbol{e} = 0, \qquad \boldsymbol{h}^2 + \boldsymbol{e}^2 = 1, \tag{40}$$

indicating that only four of the six vectorial elements are independent – the number exactly matching the four Keplerian elements involved in the definitions of h and e.

Vectorial elements possess an excellent property: they are free of all singularities related to vanishing line of nodes, zero eccentricity, polar motion or straight-line degeneracy. The non-singularity will be clearly visible in the equations of motion derived from the scaled perturbing potential

$$\mathcal{K}^* = \frac{\mathcal{K}_1}{\varepsilon^2 n_0^2 a^2} = \frac{1}{4} \left(1 - h_3^2 - e_1^2 - e_2^2 + 4e_3^2 \right),\tag{41}$$

where an additional factor $\varepsilon^{-2} n_0^{-1}$ has been introduced to change the independent variable from *t* to τ , and Keplerian elements have been expressed in terms of *e* and *h* components.

Using \mathcal{K}^* instead of V^* , we obtain from equations (38) and (39) the equations of motion:

$$h_1' = -\frac{1}{2}h_2h_3 + \frac{5}{2}e_2e_3,\tag{42}$$

$$h_2' = \frac{1}{2}h_1h_3 - \frac{5}{2}e_1e_3,\tag{43}$$

$$h'_3 = 0,$$
 (44)

$$e_1' = 2e_3h_2, (45)$$

$$e_2' = -2e_3h_1, (46)$$

$$e'_{3} = \frac{1}{2}(e_{1}h_{2} - e_{2}h_{1}). \tag{47}$$

The reader should note that equations (42)–(47) are non-singular: there are no divisors to vanish and the right-hand sides are C^{∞} . In the following discussion we will refer to the system (38)–(39) or its particular form (42)–(47) using the acronym MAC (Milankovitch– Allan–Cook) equations.

3.2 Analytical solution

Recalling the results of Section 2 and the definition of e and h, one may guess that it should be possible to solve the MAC equations in terms of elliptic functions and elliptic integrals. However, is this an easy way? Skipping equation (44), which admits an immediate solution $h_3 = \alpha$, the remaining equations can be transformed into the set of second-order equations

$$h_1'' = -\left[\frac{1}{4}(\alpha^2 - 3\beta) + 2e^2\right]h_1,$$
(48)

$$h_2'' = -\left[\frac{1}{4}(\alpha^2 - 3\beta) + 2e^2\right]h_2,$$
(49)

$$e_1'' = -(\alpha^2 - \beta - 1 + 2e^2)e_1 + 2\alpha h_1 e_3, \tag{50}$$

$$e_2'' = -(\alpha^2 - \beta - 1 + 2e^2)e_2 + 2\alpha h_2 e_3,$$
(51)

$$e_3'' = -(2\beta - \gamma)e_3 - 10e_3^3.$$
(52)

A simplification of the right-hand sides has been achieved thanks to the relations (40) and the prime integral (6) that now reads

$$\beta = e^2 - 5e_3^2 = 1 - h^2 - 5e_3^2.$$
(53)

Nevertheless, equations (48)–(52) form a set of Hill's equations with the remarkable exception of equation (52). Not only is it decoupled

from the remaining subsystem, but it is also a classical textbook creature known as the Duffing equation (Nayfeh 1973). Observe that $e_3 = es \sin \omega$, and thus knowledge of $e_3(\tau)$ will bring us closer to the knowledge of $\omega(\tau)$.

Let us inspect the new vector $q = h \times e$. Its third component is

$$q_3 = e_2 h_1 - e_1 h_2 = \sqrt{(1 - e^2)} es \cos \omega.$$
(54)

Here is our missing tile to fill the ω puzzle.

Thus we adopt the following route to solve the MAC equations. First, we solve equation (52), then we find $q_3(\tau)$, and finally we substitute all the results for e(t), $\Omega(t)$ and $\omega(t)$ into the primary definitions (36) and (37), obtaining $e(\tau)$ and $h(\tau)$.

3.2.1 Solution for e_3

Equation (52) is a Duffing equation without damping and forcing terms and its solution has the general form (Nayfeh 1973; Coppola & Rand 1990)

$$e_3 = C_3 \operatorname{cn}(B_3 \tau + \phi_0 \mid m_3), \tag{55}$$

with $0 \le m_3 < 1$. If $m_3 > 1$ equation (55) still holds true, but in such a case the inverse modulus transformation is usually preferred (Byrd & Friedman 1954). Although the qualitative study of Breiter et al. (1996) considered the Laplace vector in the nodal reference frame, their conclusions concerning e_3 and e remain valid in our case, because the axis normal to the Galactic disc is common in both cases. Thus we can set $\phi_0 = 0$ and

$$C_3 = \pm \sqrt{\frac{e_{\max}^2 - \beta}{5}} = \pm \sqrt{\frac{\xi_1 - \beta}{5}},$$
(56)

because the maximum value $e_{\max}^2 = \xi_1$ is always attained when e_3^2 is maximum (i.e. at $\omega = \pi/2$ or $\omega = 3\pi/2$). If the independent variable τ is measured from the maximum of e at $\omega = \pi/2$, we select $C_3 > 0$. The minus sign in equation (56) should be adopted if $\tau = 0$ when the maximum of e occurs at $\omega = 3\pi/2$. Note that in the circulating perihelion regime ($\beta > 0$) the choice of $\omega = \pi/2$ or $\omega = 3\pi/2$ is a matter of convention (in this paper we adopt $\omega = \pi/2$). If $\beta < 0$ and the perihelion librates, it stays close to either $\omega = \pi/2$ or $\omega = 3\pi/2$, and the choice of the sign is unique and necessary.

Differentiating twice the solution (55) we can express the lefthand side of (52) as

$$e_{3}'' = -C_{3}B_{3}(\operatorname{sn} \operatorname{dn})'$$

= $-C_{3}B_{3}^{2}\operatorname{cn} \operatorname{dn}^{2} + C_{3}B_{3}^{2}m_{3}\operatorname{sn}^{2}\operatorname{cn}$
= $C_{3}B_{3}^{2}(2m_{3} - 1)\operatorname{cn} - 2m_{3}C_{3}B_{3}^{2}\operatorname{cn}^{3}$
= $B_{3}^{2}(2m_{3} - 1)e_{3} - 2m_{3}B_{3}^{2}C_{3}^{-2}e_{3}^{3}.$ (57)

Equating the coefficients of e_3 and e_3^3 in equations (52) and (57), we obtain

$$m_3 = \frac{\xi_1 - \beta}{\xi_1 - \xi_2}, \qquad B_3 = \sqrt{\kappa},$$
 (58)

valid as long as $m_3 < 1$, i.e. for $\beta > 0$. If β is negative, the inverse modulus transformation can be used,

$$\operatorname{cn}(B_3\tau \mid m_3) = \operatorname{dn}(\sqrt{m_3} B_3\tau \mid m_3^{-1}), \tag{59}$$

and so the complete e_3 solution is

$$e_{3}(\tau) = \begin{cases} \sqrt{\frac{\xi_{1} - \beta}{5}} \operatorname{cn}(B\tau \mid m) & \text{for } \beta > 0, \\ \pm \frac{\sqrt{4 - 5\alpha^{2}}}{2\sqrt{5} \operatorname{cosh}\left[\sqrt{(4 - 5\alpha^{2})}\tau/2\right]} & \text{for } \beta = 0, \\ \pm \sqrt{\frac{\xi_{1} - \beta}{5}} \operatorname{dn}(B\tau \mid m) & \text{for } \beta < 0, \end{cases}$$
(60)

where *m* and *B* are the same as in equations (15), (16) and (17). The \pm symbol reminds us that a proper sign should be selected for those types of motion that are restricted to $e_3 > 0$ (plus sign) or $e_3 < 0$ (minus sign). For $\beta > 0$ we always select $\tau = 0$ at $\omega = \pi/2$ and hence no sign alternative is given.

3.2.2 Solution for q_3

Comparing equations (47) and (54), one spots a simple relation

$$q_3 = -2e'_3. (61)$$

Hence, knowledge of $q_3(\tau)$ can be gained thanks to a direct differentiation of equation (60) and we obtain

$$q_{3}(\tau) = \begin{cases} \frac{2}{\sqrt{5}} AB \operatorname{sn}(B\tau \mid m) \operatorname{dn}(B\tau \mid m) & \text{for } \beta > 0, \\ \pm \frac{(4 - 5\alpha^{2}) \tanh\left[\sqrt{(4 - 5\alpha^{2})} \tau/2\right]}{2\sqrt{5} \cosh\left[\sqrt{(4 - 5\alpha^{2})} \tau/2\right]} & \text{for } \beta = 0, \\ \pm \frac{2}{\sqrt{5}} A^{2} \operatorname{sn}(B\tau \mid m) \operatorname{cn}(B\tau \mid m) & \text{for } \beta < 0. \end{cases}$$
(62)

The \pm sign choice is similar to the e_3 case, and A, B and m have their usual meanings.

Owing to the fact that e, s and $\sqrt{1 - e^2}$ are all non-negative, we are able to determine the value of $\omega(\tau)$ without any ambiguity – either using a two-argument arcus tangent function of some programming languages, or resorting to the tangent of half-angle formulae:

$$\omega = 2 \arctan\left(\frac{e_3\sqrt{1-e^2}}{q_3 + e\sqrt{1-\alpha^2 - e^2}}\right) = 2 \arctan\left(\frac{-q_3 + e\sqrt{1-\alpha^2 - e^2}}{e_3\sqrt{1-e^2}}\right).$$
 (63)

3.2.3 Vectorial elements solution

Now we may substitute all the previous results into the definitions of **h** and **e** in terms of Keplerian elements. In terms of $\Omega(\tau)$, $e^2(\tau)$, $e_3(\tau)$ and $q_3(\tau)$, we obtain the momentum vector

$$h_1(\tau) = \sqrt{1 - \alpha^2 - e^2(\tau)} \sin \Omega(\tau), \tag{64}$$

$$h_2(\tau) = -\sqrt{1 - \alpha^2 - e^2(\tau)} \cos \Omega(\tau),$$
 (65)

$$h_3(\tau) = \alpha. \tag{66}$$

The first two components of the Laplace vector are

$$e_1(\tau) = \frac{q_3(\tau)\cos\Omega(\tau) - \alpha e_3(\tau)\sin\Omega(\tau)}{\sqrt{1 - \alpha^2 - e^2(\tau)}}$$
(67)

and

$$e_2(\tau) = \frac{q_3(\tau)\sin\Omega(\tau) + \alpha e_3(\tau)\cos\Omega(\tau)}{\sqrt{1 - \alpha^2 - e^2(\tau)}}.$$
(68)

It is worth noting that, if $\Omega(\tau) = 0$ is set in equations (67)–(68), we obtain the time dependence of the Laplace vector components in the nodal frame. In that frame the evolution of *e* is periodic for all $\beta \neq 0$, in agreement with the qualitative results of Breiter et al. (1996).

4 LIE-POISSON INTEGRATOR

Knowing an analytical solution is important for the understanding of phenomena related to the motion of comets in the Oort Cloud under the action of Galactic disc tides; it is indispensable if the radial tide is to be introduced as a perturbation of the disc tide. However, numerical integration may happen to be more efficient from the computational point of view. In the present section we show that equations of motion (42)–(47) can be numerically integrated by means of efficient and elegant Hamiltonian methods.

4.1 MAC equations are Hamiltonian

Interestingly, equations (38)–(39) are actually Hamiltonian equations of motion. They can be derived from a Hamiltonian V^* , being a function of variables $v = (h_1, h_2, h_3, e_1, e_2, e_3)^{\text{T}}$. 'Hamiltonian' does not mean 'canonical' in this case, because a non-canonical Lie–Poisson bracket

$$(f;g) \equiv \left(\frac{\partial f}{\partial v}\right)^{\mathrm{T}} \boldsymbol{J}(v) \frac{\partial g}{\partial v}$$
(69)

should be used to obtain equations (38)-(39) in the form

$$\dot{\boldsymbol{v}} = (\boldsymbol{v}; \, \boldsymbol{V}^*). \tag{70}$$

The structure matrix J(v) differs from the standard symplectic matrix S of the canonical formalism

$$S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$
 (71)

In our case it is

$$J(v) = \begin{pmatrix} \hat{h} & \hat{e} \\ \hat{e} & \hat{h} \end{pmatrix}, \tag{72}$$

where the 'hat map' of any vector $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ is defined as

$$\hat{\mathbf{x}} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$
(73)

helping to represent a vector product as a matrix product. For any $x, y \in \mathbb{R}^3$

$$\hat{x}y = x \times y. \tag{74}$$

4.2 Splitting method basics

Any system of differential equations can be numerically integrated by a so-called 'splitting method' if only we can split its right-hand sides into a sum of such terms that each of them generates an explicitly solvable problem. An extensive review of this technique can be found in (McLachlan & Quispel 2002). Restricting our discussion to a two-term splitting, we consider the problem

$$\dot{\mathbf{y}} = f(\mathbf{v}) = f_1(\mathbf{y}) + f_2(\mathbf{y}),$$
(75)

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where **y** is a vector of variables and f(v) is the vector of right-hand sides split into the sum of f_1 and f_2 . Let the operator $\Psi_{1,\Delta}$ stand for the solution of $\dot{y} = f_1(y)$, i.e. given the initial conditions y_0 at $t = t_0$ we obtain $y(t_0 + \Delta) = \Psi_{1,\Delta} y_0$. Similarly we define $\Psi_{2,\Delta}$ as the solution of $\dot{y} = f_2(y)$. An elementary first-order integrator is then obtained as a composition

$$\mathbf{y}(t_0 + \Delta) \simeq \Psi_{2,\Delta} \circ \Psi_{1,\Delta} \mathbf{y}_0 + O(\Delta^2).$$
(76)

It is worth noting that if the operators commute, i.e.

$$\Psi_{2,\Delta} \circ \Psi_{1,\Delta} = \Psi_{1,\Delta} \circ \Psi_{2,\Delta},\tag{77}$$

then equation (76) represents the exact solution of the full system and the error term $O(\Delta^2)$ can be dropped.

Higher-order methods result if Ψ_1 and Ψ_2 are composed with properly chosen substeps Δ . For example, the widespread Störmer– Verlet 'leapfrog' integrator is defined as

$$\mathbf{y}(t_0 + \Delta) \simeq \Psi_{1, \frac{1}{2}\Delta} \circ \Psi_{2, \Delta} \circ \Psi_{1, \frac{1}{2}\Delta} \mathbf{y}_0 + O(\Delta^3).$$
(78)

If the equations of motion possess some symmetry properties, it is desirable to split f in such a way that both f_1 and f_2 share the same properties as f. In particular, if the differential system is Hamiltonian, with $f = (y; \mathcal{H})$, the logical approach is to obtain f_1 and f_2 from two components of the Hamiltonian function $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$. Then we have $f_1 = (y; \mathcal{H}_1)$ and $f_2 = (y; \mathcal{H}_2)$. In these circumstances, the resulting maps Ψ_1 and Ψ_2 will represent Hamiltonian motions and the same will hold true for their composition. This strategy leads to well-known 'symplectic integrators' if the Poisson bracket is canonical Yoshida (1993). Obviously it is also valid for more complicated Poisson brackets: an example can be found in Touma & Wisdom (1994). In order to apply a splitting method to the problem of Galactic disc perturbations, one should find the two maps Ψ_1 and Ψ_2 by solving the equations of motion for the variables v with the Lie–Poisson bracket (72) and partitioned Hamiltonian function (41). Obviously, any constant term in the Hamiltonian function has no influence on the equations of motion, so we replace \mathcal{K}^* by

$$\mathcal{H} = \mathcal{K}^* - \frac{1}{4} = \frac{1}{4} \left(-h_3^2 - e_1^2 - e_2^2 + 4e_3^2 \right) = \mathcal{H}_1 + \mathcal{H}_2.$$
(79)

We propose the following partition:

$$\mathcal{H}_1 = -\frac{1}{4} \left(e_1^2 + e_2^2 + e_3^2 \right) = -\frac{1}{4} \boldsymbol{e}^2, \tag{80}$$

$$\mathcal{H}_2 = \frac{1}{4} \left(5e_3^2 - h_3^2 \right). \tag{81}$$

As we will show below, both terms define problems easily solvable in terms of elementary functions.

4.3 Motion induced by \mathcal{H}_1

Let us solve

$$\boldsymbol{v}' = (\boldsymbol{v}; \mathcal{H}_1) = -\frac{1}{4} \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{v}} \right)^{\mathrm{T}} \boldsymbol{J}(\boldsymbol{v}) \frac{\partial(\boldsymbol{e}^2)}{\partial \boldsymbol{v}}$$
$$= -\frac{1}{2} \begin{pmatrix} \hat{\boldsymbol{h}} & \hat{\boldsymbol{e}} \\ \hat{\boldsymbol{e}} & \hat{\boldsymbol{h}} \end{pmatrix} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{e} \end{pmatrix}.$$
(82)

As we see, \mathcal{H}_1 generates the equations of rotation around the fixed vector *h*

$$\boldsymbol{h}' = \boldsymbol{0},\tag{83}$$

$$\boldsymbol{e}' = -\frac{1}{2}\,\boldsymbol{h}\times\boldsymbol{e}.\tag{84}$$

Hence, the solution of equation (82) takes the simple matrix form

$$\Psi_{1,\Delta} \boldsymbol{v}_0 = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_1(\boldsymbol{h}_0, \Delta) \end{pmatrix} \begin{pmatrix} \boldsymbol{h}_0 \\ \boldsymbol{e}_0 \end{pmatrix}, \tag{85}$$

where, according to the Euler–Rodrigues formula (Marsden & Ratiu 1999)

$$\boldsymbol{M}_{1}(\boldsymbol{h},\Delta) = \boldsymbol{I} + \frac{\sin\psi_{1}}{h}\,\hat{\boldsymbol{h}} + 2\frac{\sin^{2}\left(\psi_{1}/2\right)}{h^{2}}\,\hat{\boldsymbol{h}}\hat{\boldsymbol{h}},\tag{86}$$

and the rotation angle is

$$\psi_1 = -\frac{1}{2}h\Delta. \tag{87}$$

From the geometrical point of view, this part of the Hamiltonian induces the rotation of the Laplace vector along the orbital plane.

4.4 Motion induced by \mathcal{H}_2

In our opinion the most instructive way to solve $v' = (v; \mathcal{H}_2)$ is to take an indirect path, partitioning the Hamiltonian (81) into $\mathcal{H}_2 = \mathcal{H}_{21} + \mathcal{H}_{22}$, where

$$\mathcal{H}_{21} = -\frac{1}{4}h_3^2, \qquad \mathcal{H}_{22} = \frac{5}{4}e_3^2.$$
 (88)

The equations of motion generated by \mathcal{H}_{21} are

$$\boldsymbol{h}' = \frac{1}{2}(0, 0, h_3)^{\mathrm{T}} \times \boldsymbol{h}, \tag{89}$$

$$\mathbf{e}' = \frac{1}{2}(0, 0, h_3)^{\mathrm{T}} \times \mathbf{e}.$$
(90)

In other words, \mathcal{H}_{21} rotates the orbital plane around the Oz axis and so

$$\Psi_{21,\Delta} \boldsymbol{v}_0 = \begin{pmatrix} \boldsymbol{M}_{21}(\boldsymbol{h}_0, \Delta) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_{21}(\boldsymbol{h}_0, \Delta) \end{pmatrix} \begin{pmatrix} \boldsymbol{h}_0 \\ \boldsymbol{e}_0 \end{pmatrix}, \qquad (91)$$

where

$$\boldsymbol{M}_{21}(\boldsymbol{h},\Delta) = \begin{pmatrix} \cos\psi_{21} & -\sin\psi_{21} & 0\\ \sin\psi_{21} & \cos\psi_{21} & 0\\ 0 & 0 & 1 \end{pmatrix},$$
(92)

and $\psi_{21} = 1/2h_3\Delta$.

The second part of the Hamiltonian functions leads to

$$\mathbf{h}' = (\mathbf{h}; \mathcal{H}_{22}) = -\frac{5}{2}(0, 0, e_3)^{\mathrm{T}} \times \mathbf{e},$$
 (93)

$$e' = (e; \mathcal{H}_{22}) = -\frac{5}{2}(0, 0, e_3)^{\mathrm{T}} \times h.$$
 (94)

For these equations we find no obvious geometrical meaning; yet, they form a linear system admitting a simple solution

$$\Psi_{22,\Delta} \boldsymbol{v}_0 = \begin{pmatrix} \boldsymbol{M}_{22}(\boldsymbol{e}_0, \Delta) & \boldsymbol{N}_{22}(\boldsymbol{e}_0, \Delta) \\ \boldsymbol{N}_{22}(\boldsymbol{e}_0, \Delta) & \boldsymbol{M}_{22}(\boldsymbol{e}_0, \Delta) \end{pmatrix} \begin{pmatrix} \boldsymbol{h}_0 \\ \boldsymbol{e}_0 \end{pmatrix},$$
(95)

where

$$\boldsymbol{M}_{22}(\boldsymbol{e}, \Delta) = \begin{pmatrix} \cos\psi_{22} & 0 & 0\\ 0 & \cos\psi_{22} & 0\\ 0 & 0 & 1 \end{pmatrix},$$
(96)

$$N_{22}(\boldsymbol{e}, \Delta) = \begin{pmatrix} 0 & \sin\psi_{22} & 0\\ -\sin\psi_{22} & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \qquad (97)$$

and $\psi_{22} = 5/2e_3\Delta$. Note that only this map affects the lengths of both **h** and **e**, although the length of ||v|| = 1 remains intact due

to the properties of the Lie–Poisson bracket (69) with the structure matrix (72).

One can observe that the two maps commute

$$\Psi_{21,\Delta} \circ \Psi_{22,\Delta} \boldsymbol{v}_0 = \Psi_{22,\Delta} \circ \Psi_{21,\Delta} \boldsymbol{v}_0.$$
(98)

This fact could be expected from the beginning, because $(\mathcal{H}_{21}; \mathcal{H}_{22}) = 0$, but one can verify it easily by evaluating the products of the two matrices present in (91) and (95). According to the remark made after equation (76), this means that the exact solution of the problem with the Hamiltonian \mathcal{H}_2 is $\Psi_2 = \Psi_{21} \circ \Psi_{22}$ or, in full,

$$\Psi_{2,\Delta} \boldsymbol{v}_0 = \begin{pmatrix} \boldsymbol{M}^* & \boldsymbol{N}^* \\ \boldsymbol{N}^* & \boldsymbol{M}^* \end{pmatrix} \boldsymbol{v}_0, \tag{99}$$

where

$$\boldsymbol{M}^* = \begin{pmatrix} c_{21}c_{22} & -s_{21}c_{22} & 0\\ s_{21}c_{22} & c_{21}c_{22} & 0\\ 0 & 0 & 1 \end{pmatrix},$$
(100)

$$N^* = \begin{pmatrix} s_{21}s_{22} & c_{21}s_{22} & 0\\ -c_{21}s_{22} & s_{21}s_{22} & 0\\ 0 & 0 & 0 \end{pmatrix},$$
(101)

and $s_{ij} = \sin \psi_{ij}, c_{ij} = \cos \psi_{ij}$.

4.5 General remarks

Using the two building blocks Ψ_1 and Ψ_2 , defined in equations (85) and (99), one is able to construct at will any preferred high-order integrator. Recipes for high-order splitting method integrators and more references can be found in Tsitouras (1999) or McLachlan & Quispel (2002). All methods, however, will have the following properties:

(i) the prime integral h_3 = constant is conserved exactly (actually the equation for h'_3 can be omitted);

(ii) the geometric prime integrals $e^2 + h^2 = 1$ and $h \cdot e = 0$ will be conserved up to the computer round-off errors;

(iii) the Hamiltonian function \mathcal{H} will have no secular error except for the round-off effects; the oscillations of \mathcal{H} will have an amplitude depending on the integration step.

5 CONCLUSIONS

The main goal of the present work was to clear the path towards an analytical treatment of cometary perturbations due to the complete Galactic tides model. For these reasons we have tied the loose ends of the Matese–Whitman theory that were apparently considered unimportant in earlier works. In principle, the results concerning the lines of nodes and apsides could be obtained without introducing the vectorial elements and their equations of motion; however, the MAC equations are so simple and elegant that, we believe, they supersede the Lagrange 'planetary' equations or canonical equations in both accuracy and speed. All their right-hand sides are simple quadratic forms and can be easily differentiated without the occurrence of any singularity. We have presented an explicit Lie–Poisson integrator for the MAC equations, but other 'general-purpose' numerical integrators can be used as well. The simplicity of the right-hand sides offers an opportunity to use high-order Taylor series methods. It is also possible that constant step Gauss–Legendre (or Runge–Kutta–Butcher) integrators are the perfect match for this particular problem: they will conserve all the integrals of motion including the Hamiltonian, because these methods by definition conserve quadratic integrals (Sanz-Serna & Calvo 1994).

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REFERENCES

- Abramowitz M., Stegun I. A., 1972, Handbook of Mathematical Functions. Wiley-Interscience, New York
- Allan R. R., Cook G. E., 1964, Proc. R. Soc. London A, 280, 97
- Allan R. R., Ward G. N., 1963, Proc. Camb. Phil. Soc., 59, 669
- Brasser R., 2001, MNRAS, 324, 1109
- Breiter S., Dybczyński P., Elipe A., 1996, A&A, 315, 618
- Brouwer D., Clemence G., 1961, Methods of Celestial Mechanics. Academic Press, New York
- Byl J., 1983, Moon Planets, 29, 121
- Byrd P. F., Friedman M. D., 1954, Handbook of Elliptic Integrals for Engineers and Physicists. Springer, Berlin
- Chebotarev G. A., 1966, SvA, 10, 341
- Coppola V. T., Rand R. H., 1990, Acta Mech., 81, 125
- Fouchard M., 2004, MNRAS, 349, 347
- Heisler J., Tremaine S., 1986, Icarus, 65, 13
- McLachlan R. I., Quispel G. R. W., 2002, Acta Num., 11, 341
- Marsden J. E., Ratiu T. S., 1999, Introduction to Mechanics and Symmetry (Texts in Applied Mathematics, Vol. 17). Springer, New York
- Matese J. J., Whitman P. G., 1989, Icarus, 82, 389
- Matese J. J., Whitman P. G., 1992, Celest. Mech. Dyn. Astron., 54, 13
- Milankovitch M., 1939, Acad. Serb. Bull. Acad. Sci. Mat. Nat. A, 6, 1
- Musen P., 1961, J. Geophys. Res., 66, 1659
- Nayfeh A. H., 1973, Perturbation Methods. Wiley, New York
- Oort J. H., 1950, Bull. Astron. Inst. Neth., 11, 91
- Sanz-Serna J. M., Calvo M. P., 1994, Numerical Hamiltonian Problems. Chapman and Hall, London
- Touma J., Wisdom J., 1994, AJ, 107, 1189
- Tsitouras C., 1999, Celest. Mech. Dyn. Astron., 74, 223
- Yoshida H., 1993, Celest. Mech. Dyn. Astron., 56, 27

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