DOUBLE MATERIAL SEGMENT AS THE MODEL OF IRREGULAR BODIES

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(Received: 4 January 2002; revised: 3 September 2002; accepted: 17 October 2002)

Abstract. We propose a new, simple model to describe the gravity field of irregular, nonspherical celestial bodies, like small moons or minor asteroids. The simple idea of Duboshin to use a material straight segment for such bodies is extended by combining two perpendicular segments of different lengths and masses. In typical situations, when the longest axis of the body coincides with one segment, the remaining segment must have an imaginary length. The potential remains a real function even if one segment is imaginary. The new model is confronted with the exact form of an ellipsoid's potential and with two alternative simple models for a two-axial and a three-axial ellipsoid.

Key words: potential, gravity field models, small bodies

1. Introduction

Space missions to comets and asteroids as well as discoveries of binary asteroids have brought back and old problem: how to approximate efficiently the potential of irregular bodies? A standard model of a three-axial homogeneous ellipsoid (Scheeres, 1994) is a solid answer to this question, but its mathematical form, requiring elliptic integrals, is not computationally cheap and too complicated for analytical considerations. At this point, we should specify explicitly, that the notion of an 'irregular body' in the present paper is quite special: it refers to cigar-shaped, but still ellipsoid-resembling objects. There exist much more irregular asteroids like 4769 Castalia or 216 Kleopatra. In those cases, a polyhedron representation developed by Werner and Scheeres (1997) is the best solution. The polyhedron model, although robust and accurate, will not be discussed in the present paper: it is just too sophisticated for the simple 'irregular bodies' considered here, because it involves too many free parameters. A similar argument applies to other kinds of many-parameters approximations like mascons.

Let us return to the ellipsoid approximation. Expanding the potential of an ellipsoid into the Legendre series (Balmino, 1994) provides a form well suited for an analytical treatment, but the series are lengthy and the radius of their absolute convergence is $\bar{r} = \sqrt{a^2 - c^2}$, where a and c are the major and the least semi-axes. Only for the ellipsoids with $a/c \leqslant \sqrt{2}$ the Legendre series may uniformly approximate the potential down to the ellipsoid's surface (Balmino, 1994).

Celestial Mechanics and Dynamical Astronomy **86:** 131–141, 2003.

Trading accuracy for the convenience, some authors proposed simplified models. Scheeres and Hu (2001) considered a truncated Legendre expansion with $C_{2,0}$ and $C_{2,2}$ harmonics. Another simple model originating from the idea of Duboshin (1959) has also been recently discussed (Riaguas et al., 1999, Riaguas, 2000); the model replaces a body with a material straight segment aligned with its major axis.

In the present paper we extend the model of Duboshin–Riaguas, combining two perpendicular segments. Our double segment model is then compared with other approximations of a three-axial ellipsoid.

2. The Double Segment Model

Let us start with a classical formula for the Duboshin–Riaguas (DR) potential. According to Duboshin (1959) and Riaguas et al. (1999), an infinitely thin material segment of length $2l$ and mass m, placed along the axis Ox, such that O coincides with its centre of mass, has a potential

$$V_1 = -\frac{\mu}{2l} \ln\left(\frac{s+2l}{s-2l}\right), \tag{1}$$

where $\mu = k^2 m$ (with k – Gaussian gravity constant),

$$s = \sqrt{r^2 + 2xl + l^2} + \sqrt{r^2 - 2xl + l^2}, \tag{2}$$

and $\mathbf{r} = (x, y, z)^{\mathrm{T}}$ is the radius vector of a given point outside the segment. This potential is intended to represent an elongated ellipsoid with the major axis $2a$ aligned with the axis Ox and the centre of mass at O. Due to the axial symmetry of the segment, the DR model should formally be restricted to spheroids with $a > b = c$.

In order to break the symmetry of the DR model let us introduce a second segment, perpendicular to the first one. After checking all three possible configurations, we have decided to combine the segments oriented along the axes Ox and Oz.[1] The segments have lengths $2l_1$ and $2l_3$, respectively, and their masses m_1 and m_3 can be different. Extending the formula of Equation (1), we can write down the potential of the double segment as

$$V_{13} = -\frac{\mu_1}{2l_1} \ln\left(\frac{s_1+2l_1}{s_1-2l_1}\right) - \frac{\mu_3}{2l_3} \ln\left(\frac{s_3+2l_3}{s_3-2l_3}\right), \tag{3}$$

where

$$s_1 = \sqrt{r^2 + 2xl_1 + l_1^2} + \sqrt{r^2 - 2xl_1 + l_1^2}, \tag{4}$$

[1] This '13' model has been selected because it led to the best results for cigar-shaped bodies like the ones discussed in Section 4. A brief account of alternative combinations is given in Section 5.

$$s_3 = \sqrt{r^2 + 2zl_3 + l_3^2} + \sqrt{r^2 - 2zl_3 + l_3^2}, \tag{5}$$

and $\mu_1 = k^2 m_1$, $\mu_3 = k^2 m_3$. The sum of the masses

$$m = m_1 + m_3, \tag{6}$$

is assumed to be equal to the mass of the modelled ellipsoid.

The potential V_{13} is relatively simple and its gradient can be easily derived, so that equations of motion for a negligible mass particle in the field of the double segment take the form

$$\ddot{x} = -\frac{\partial V_{13}}{\partial x} = -2\mu x \left[\frac{1-\kappa}{s_1 p_1} + \frac{\kappa s_3}{p_3(s_3^2 - 4l_3^2)} \right], \tag{7}$$

$$\ddot{y} = -\frac{\partial V_{13}}{\partial y} = -2\mu y \left[\frac{(1-\kappa)s_1}{p_1(s_1^2 - 4l_1^2)} + \frac{\kappa s_3}{p_3(s_3^2 - 4l_3^2)} \right], \tag{8}$$

$$\ddot{z} = -\frac{\partial V_{13}}{\partial z} = -2\mu z \left[\frac{(1-\kappa)s_1}{p_1(s_1^2 - 4l_1^2)} + \frac{\kappa}{s_3 p_3} \right], \tag{9}$$

where

$$p_1 = \sqrt{r^2 + 2xl_1 + l_1^2}\sqrt{r^2 - 2xl_1 + l_1^2}, \tag{10}$$

$$p_3 = \sqrt{r^2 + 2zl_3 + l_3^2}\sqrt{r^2 - 2zl_3 + l_3^2}, \tag{11}$$

and a mass ratio parameter κ have been introduced

$$\kappa = \frac{m_3}{m}. \tag{12}$$

Once the model is formulated, we may focus on the question how to determine its parameters.

3. Parameters of the Model

Throughout this paper we discuss our model V_{13} as an approximation of a homogeneous ellipsoid's potential V_e. The latter has a form (MacMillan, 1930)

$$V_\mathrm{e} = -\frac{3}{4}\mu \int_\sigma^\infty Q(x, y, z, a, b, c, s)\,\mathrm{d}s, \tag{13}$$

$$Q = \frac{1 - x^2/(a^2+s) - y^2/(b^2+s) - z^2/(c^2+s)}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}, \tag{14}$$

where σ is the positive root of $Q = 0$. Given a point (x, y, z) outside the ellipsoid, the equation $Q = 0$, with s treated as the unknown, has only one positive real root, thus σ is uniquely defined (MacMillan, 1930).

Aiming at giving the proper values to the three parameters l_1, l_3, and κ, we may consider two different strategies. The most simple approach refers to the matching of Legendre series expansions of V_{13} and V_{e}. Another, more tedious variant requires the least squares adjustment on a grid of points covering the surface of the ellipsoid.

3.1. ASYMPTOTIC ADJUSTMENT

According to the results obtained by Duboshin (1959) for the potential V_1, we can write down the Legendre series for V_{13} in the following, simple form

$$V_{13} = -\frac{\mu}{r} \sum_{n=0}^{\infty} \frac{(1-\kappa) P_{2n}(x/r) l_1^{2n} + \kappa P_{2n}(z/r) l_3^{2n}}{(2n+1) r^{2n}}, \tag{15}$$

where $P_\nu(\xi)$ stands for the degree ν Legendre polynomial of the variable ξ. The leading terms of Equation (15) are

$$V_{13} \approx -\frac{\mu}{r} \left[1 - \frac{(1-\kappa) l_1^2 + \kappa l_3^2}{6r^2} + \frac{((1-\kappa) l_1^2 x^2 + \kappa l_3^2 z^2)}{2r^2} \right]. \tag{16}$$

Equation (16) can be compared with the leading terms of the classical spherical harmonics representation of a potential of an arbitrary body contained within a sphere of radius a

$$V \approx -\frac{\mu}{r} \left[1 - \frac{a^2 (C_{2,0} + 6C_{2,2})}{2r^2} + \frac{6a^2 C_{2,2} x^2}{r^4} + \frac{3}{2} \frac{a^2 (C_{2,0} + 2C_{2,2}) z^2}{r^4} \right]. \tag{17}$$

According to Balmino (1994), the coefficients for a homogeneous ellipsoid are

$$C_{2,0} = \frac{2c^2 - a^2 - b^2}{10a^2}, \qquad C_{2,2} = \frac{a^2 - b^2}{20a^2}, \tag{18}$$

and thus we obtain

$$l_1 = 2a \sqrt{\frac{3C_{2,2}}{1-\kappa}} = \sqrt{\frac{3(a^2 - b^2)}{5(1-\kappa)}}, \tag{19}$$

$$l_3 = a \sqrt{\frac{3C_{2,0} + 6C_{2,2}}{\kappa}} = \sqrt{\frac{3(c^2 - b^2)}{5\kappa}}. \tag{20}$$

Equation (20) leads us to the important conclusion, that the length l_3 has to be assumed an imaginary quantity.

3.2. REFORMULATION FOR THE IMAGINARY SEGMENT

Even with the segment of an imaginary length, the potential V_{13} remains real-valued. Indeed, after substituting

$$l_3 = iL_3, \tag{21}$$

we can compute $V_{1,3}$ without using the complex arithmetics. After some elementary operations on complex conjugate quantities, we obtain

$$V_{13} = (1-\kappa)V_1(l_1) - \frac{\kappa\mu}{L_3}\arctan\left(\frac{2L_3}{s_3}\right), \tag{22}$$

where

$$s_3 = \sqrt{2(r^2 - L_3^2 + p_3)}, \tag{23}$$

$$p_3 = \sqrt{(r^2 - L_3^2)^2 + 4z^2L_3^2}. \tag{24}$$

The expressions for the partial derivatives of V_{13} with the imaginary l_3 also remain simple: we can still use Equations (7)–(9), provided the formulas (23) and (24) are applied and $l_3^2 = -L_3^2$ is substituted.

3.3. LEAST SQUARES FIT

The asymptotic adjustment of the parameters is simple, but it also has two inconvenient aspects: it is only indirectly related to the values of the gravity field close to the surface of an ellipsoid, and it gives no hint about the appropriate value of κ. In the alternative approach, we have considered a nonlinear least squares adjustment of the parameters l_1, L_3, and κ on a grid of sample points covering the ellipsoid's surface. Due to the symmetries of the problem, we have distributed 330 of sample points $\mathbf{r}_n$ along one octant with $x \geqslant 0$, $y \geqslant 0$, and $z \geqslant 0$. The set of sample points was generated as a sum three subsets

$$\mathbf{r}_n \in S_1 \cup S_2 \cup S_3 \tag{25}$$

with

$$S_1 = \{\mathbf{r}_{j,q} : (a\cos(j\delta)\cos(q\delta), b\cos(j\delta)\sin(q\delta), c\sin(j\delta))^{\mathrm{T}}\},$$

$$S_2 = \{\mathbf{r}_{j,q} : (a\sin(j\delta), b\cos(j\delta)\sin(q\delta), c\cos(j\delta)\cos(q\delta))^{\mathrm{T}}\},$$

$$S_3 = \{\mathbf{r}_{j,q} : (a\cos(j\delta)\sin(q\delta), b\sin(j\delta), c\cos(j\delta)\cos(q\delta))^{\mathrm{T}}\},$$

where $j = 0, \ldots, 9, q = 0, \ldots, 10$ in all subsets and $\delta = \pi/20$. This kind of sampling covers the surface almost uniformly.

The χ^2 merit function, defined as

$$\chi^2 = \frac{a^2}{\mu}\sum_{n=1}^{330}([V_\mathrm{e} - V_{13}]_{\mathbf{r}_n})^2, \tag{26}$$

has been evaluated on the grid and minimised by means of the Levenberg–Marquardt algorithm. All computations have been performed using the `NonliearFit` procedure of *Mathematica* (Adamchik et al., 1993).

4. Comparison of Models

In order to check the quality of the proposed improvement, we have performed a comparison of three models with two test bodies: a spheroidal approximation of the asteroid 243 Ida, and an ellipsoid representing the Mars satellite Phobos. For both bodies we have checked the behaviour of various approximate potentials along the Ox, Oy, and Oz axes. Throughout this section we will use the following abbreviations: P2 – the Legendre expansion including the $C_{2,0}$ and $C_{2,2}$ harmonic coefficients, DR – the single material segment model of Duboshin and Riaguas, BB – the present model of two material segments. The P2 potential (Scheeres and Hu, 2001) is given by Equation (17).

4.1. ELONGATED SPHEROID CASE

The first test case is an elongated spheroid, whose semi-axes ratios reflect the shape of 243 Ida, namely

$$\frac{b}{a} = \frac{c}{a} = 0.39655. \tag{27}$$

The semi-axes ratio for this body is significantly less than the limiting value $1/\sqrt{2} \approx 0.707$ required for the uniform and absolute convergence of Legendre series. One should expect that the P2 model with asymptotic parameters will obviously have a low accuracy in the vicinity of the ellipsoid. From the purely formal standpoint, the $(a/r)^n$ Legendre series approximation is no longer valid for $r \leqslant \bar{r} = \sqrt{a^2 - c^2} \approx 0.92$. However, we were interested in observing the behaviour of the standard P2 model even inside the limit sphere $r = \bar{r}$. Discussing the truncated Legendre series we meet no singularities when crossing the sphere of convergence and thus, unlike the infinite series, the P2 model should only gradually degrade with the decreasing values of r. Although the coefficients $C_{2,0}$ and $C_{2,2}$ no longer have correct values, the question of how much can we improve P2 through the adjustment of parameters is worth answering.

Table I provides all asymptotic and adjusted parameters of the three studied models. Figure 1 presents the relative errors of the studied potentials with respect to the ellipsoid potential V_{e},

$$\delta U = 1 - \frac{U}{V_{\mathrm{e}}}, \tag{28}$$

where U stands for the potential of the P2, DR, or BB model. The unit of length along the horizontal axes is a.

TABLE I

Parameters of the discussed models for two test bodies[a]

	243 Ida		Phobos	
	Asymptotic	Adjusted	Asymptotic	Adjusted
$C_{2,0}$ for P2	−0.08427	−0.0436 (0.0020)	−0.06712	−0.0612 (0.0008)
$C_{2,2}$ for P2	0.04214	0.0218 (0.0007)	0.018	0.0166 (0.0003)
l_1 for DR	0.71109	0.7073 (0.0018)	0.464758	0.5128 (0.0054)
l_1 for BB	0.71109	0.7668 (0.0015)	0.65727	0.5981 (0.0056)
L_3 for BB	0	0.074 (0.020)	0.43209	0.4760 (0.0073)
κ for BB	0	0.1360 (0.0029)	0.5	0.383 (0.012)

[a] Numbers in brackets are the standard errors.

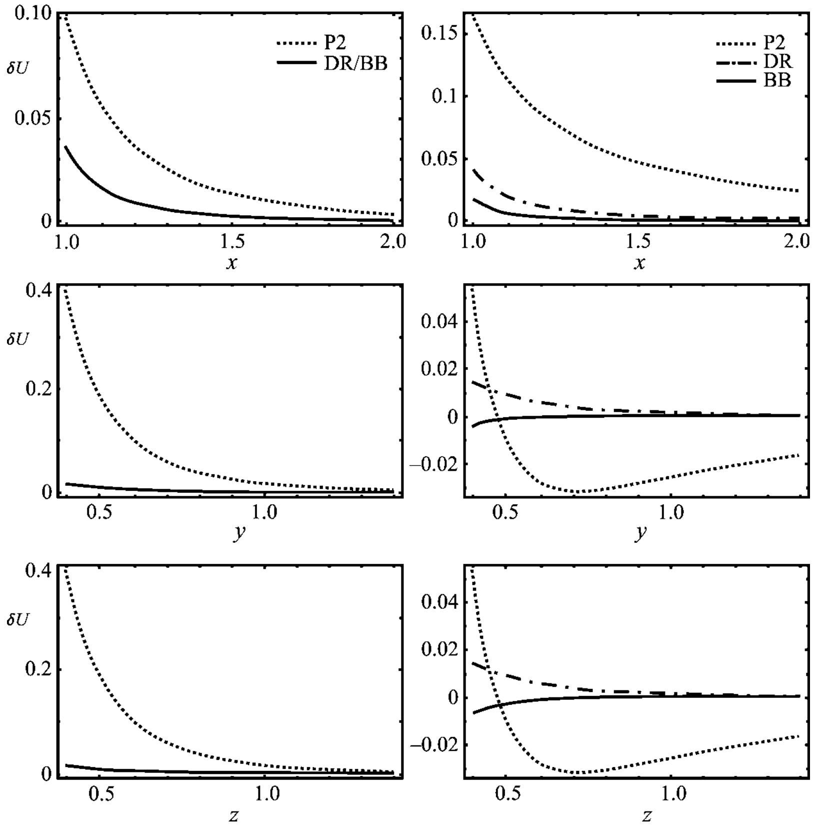

Figure 1. Relative errors of various potential models as functions of distance along three axes for asymptotic parameters (left) and the least squares adjusted parameters (right). The case of the 243 Ida spheroid.

The three plots in the left column of Figure 1 present the errors of the studied potentials with the asymptotic values of parameters. The values of $C_{2,0}$ and $C_{2,2}$ have been taken from Equation (18). Assuming $\kappa = 0$, we have computed the length l_1 from Equation (19). As it follows from Equation (20), the length $l_3 = 0$ for the $b = c$ case, and thus the DR and BB models are equivalent as far as a spheroid with the Ox axial symmetry is concerned. Without an adjustment, the DR/BB model is significantly better than P2, especially along the Oy and Oz axes, where the distance can be smaller than $\sqrt{a^2 - c^2}$. Note, that DR/BB perform better even at the distance $r > a$, where the Legendre expansion is properly defined.

The right column of Figure 1 shows how much can a model be improved due to the adjustment of parameters. For the P2 model, the major part of the improvement is spent on reducing the Oy and Oz errors; actually, it is achieved at the expense of the accuracy along the Ox axis. As far as the DR model is concerned, there is no substantial gain in its accuracy due to the adjustment of l_1. The BB model gains factor 2 in accuracy, but one may doubt if this gain is worth destroying the axial symmetry through the introduction of a second segment. The quality of each model has been reflected in the values of χ^2 obtained at the end of the adjustments: 9.5 for P2, 0.24 for DR, and 0.04 for BB.

4.2. THREE-AXIAL ELLIPSOID CASE

The ellipsoid that represents Phobos has the axes ratios

$$\frac{b}{a} = 0.8, \qquad \frac{c}{a} = 0.696. \tag{29}$$

This time c/a is only slightly smaller than $1/\sqrt{2}$ and from the formal standpoint the Legendre series are still not valid close to the ellipsoid's surface in polar regions. The results of comparison are shown in Figure 2, arranged similarly to Figure 1. For the asymptotic values of the parameters (Table I) the BB model is the best only along the Oz axis, whereas P2 is slightly better along the Ox and Oy axes. It is not surprising, that P2 behaves better than the spheroidal DR. The effect of the adjustment turns the BB model into the best one, because the high Oz accuracy with the asymptotic constant can be traded to gain in both the remaining directions. On the other hand, we observe a significant Ox error growth of P2 in order to improve the Oy and Oz accuracy. The values of χ^2 imply the following arrangement of models in this test case: DR is the worst, with $\chi^2 = 0.24$, then comes P2 with $\chi^2 = 0.07$, and BB with the smallest $\chi^2 = 0.006$.

In order to give readers an idea of how the error of BB behaves in three dimensions, Figure 3 provides two sections of the error level surfaces: in the Oxy plane to the left and in the Oyz plane to the right.

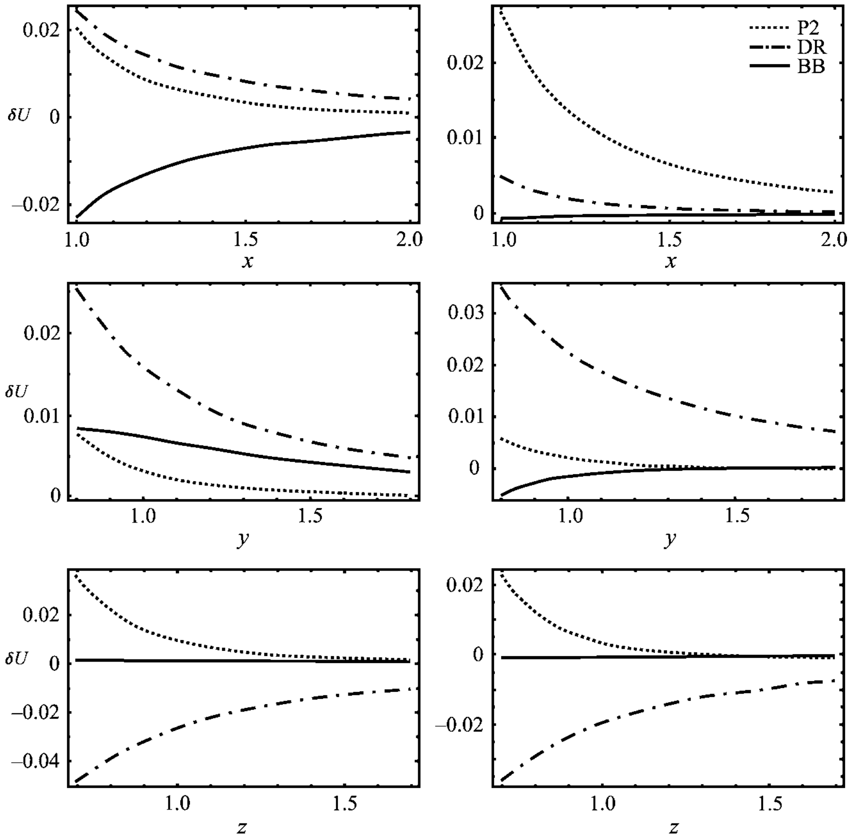

Figure 2. Relative errors of various potential models as functions of distance along three axes for asymptotic parameters (left) and the least squares adjusted parameters (right). The case of the Phobos ellipsoid.

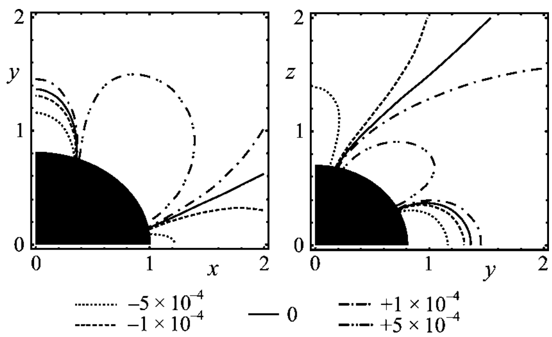

Figure 3. The contours of equal relative errors of the adjusted two segments model for the Phobos case.

5. Conclusions

The model of two material segments seems to be an efficient tool, that is both computationally cheap and fairly accurate. It can be applied in the problem of orbital motion around asteroids and small moons, especially in the situations where orbiting bodies approach the surface of a primary (e.g. the evolution of dust after a meteoroid impact). The quality of the model with adjusted parameters is much better visible from the global values of χ^2 than from the plots along three sample directions (Figures 1 and 2).

The paper has focused on the xz variant of the double segment model. We have also tested two alternative cases: xy and yz. In the xy variant, where the second segment is always real, we found that occasionally the Oy segment, can be longer than the intermediate axis b of the approximated ellipsoid. This is quite inconvenient and leads to singularities. Even in nonsingular cases, the accuracy of approximation was worse than for the xz model. The yz model looks artificial at the first glimpse; indeed it behaves much worse than the remaining two.

Although we have only discussed the case where the semi-axis a is always greater than b, one may apply the model of two material segments in other situations. For example, a Jupiter family planet with $a = b > c$ can be modelled by assuming $l_1 = 0$ and an imaginary value of l_3.

Acknowledgements

The reviewers comments about the preliminary version of the paper were very valuable and helped us much to improve the text. The work has been supported by the Polish Committee of Research (KBN) grant 2 PO3D 007 18.

References

Adamchik, V., Adams, J., Bocharov, A., Boyland, P., Chandra, A., He, Y., Keiper, J., Marichev, O., Martin, E., Novak, J., Petkovsek, M., Skiena, S., Vardi, I., Wenzlow, A., Wickham-Jones, T. and Withoff, D.: 1993, *Guide to Standard Mathematica Packages*. Wolfram Research, Champaign.

Balmino, G.: 1994, 'Gravitational potential harmonics from the shape of an homogeneous body'. *Celest. Mech. Dyn. Astron.* **60**, 331–364.

Duboshin, G. N.: 1959, 'On one particular case of the problem of the translational–rotational motion of two bodies'. *SvA* **3**, 154–165.

MacMillan, W. D.: 1930, *The Theory of the Potential*. McGraw-Hill, New York.

Riaguas, A.: 2000, 'Dinámica Orbital Alrededor de Cuerpos Celestes con Forma Irregular'. PhD thesis, Universidad de Zaragoza.

Riaguas, A., Elipe, A. and Lara, M.: 1999, 'Periodic orbits around a massive straight segment'. *Celest. Mech. Dyn. Astron.* **73**, 169–178.

Scheeres, D. J.: 1994, 'Dynamics about uniformly rotating triaxial ellipsoids: Application to asteroids'. *Icarus* **121**, 67–87.

Scheeres, D. J. and Hu, W.: 2001, 'Secular motion in a 2nd degree and order gravity field with no rotation'. *Celest. Mech. Dyn. Astron.* **79**, 183–200.

Werner, R. A. and Scheeres, D. J.: 1997, 'Exterior gravitation of a polyhedron derived and compared with harmonic and mascon gravitation representations of asteroid 4769 Castalia'. *Celest. Mech. Dyn. Astron.* **65**, 313–344.