EXTENDED FUNDAMENTAL MODEL OF RESONANCE

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Abstract. Fundamental models are the simplest, one degree of freedom Hamiltonians that serve as a tool to understand the qualitative effects of various resonances. A new, extended fundamental model (EFM) is proposed in order to improve the classical, Andoyer type, second fundamental model (SFM). The EFM Hamiltonian differs from the SFM by the addition of a term with the third power of momentum; it depends on two free parameters. The new model is studied for the case of a first-order resonance, where up to five critical points can be present. Similarly, to the respective SFM, it admits only the saddle-node bifurcations of critical points, but its advantage lies in the capability of generating the separatrix bifurcations, known also as saddle connections. The reduction of parameters for the EFM has been performed in a way that allows the use of the model in the case of the so-called abnormal resonance.

Key words: resonance, fundamental models, analytical methods

1. Introduction

1.1. THE NOTION OF A FUNDAMENTAL MODEL

Let us consider a Hamiltonian system with N degrees of freedom subject to a resonance. In the most simple of nontrivial cases, a resonance inhibits the normalization procedure for this system, so that only $N-1$ angles can be eliminated by means of a standard Lie transformation (Ferraz-Mello, 1988) and we are left with a partially normalized, one degree of freedom Hamiltonian

$$\mathcal{H} = \mathcal{H}_0(I) + \mathcal{H}_1(I, \phi), \tag{1}$$

generating the equations of motion

$$\dot{I} = -\frac{\partial \mathcal{H}_1}{\partial \phi}, \qquad \dot{\phi} = \frac{\partial \mathcal{H}_0}{\partial I} + \frac{\partial \mathcal{H}_1}{\partial I}. \tag{2}$$

Equations (1) and (2) are obviously too general to provide useful results, whereas the expression of $\mathcal{H}$ in some particular case may be too cumbersome and overloaded with a multitude of parameters.

The notion of a fundamental model of resonance emerged in the search of a compromise between the general and particular aspects of the problem. Let us

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define the fundamental model of resonance as a one degree of freedom Hamiltonian $\mathcal{K}$, having the simplest possible form and depending on the minimum number of parameters, but still capable of reproducing the qualitative features of a given reduced Hamiltonian $\mathcal{H}$. The present study is restricted to the resonant problems where $\mathcal{H}_1$ can be approximated as a single term $2m\pi$-periodic with respect to the angle ϕ. Throughout the paper m will be named the resonance order. The three known fundamental models concerning this case will be briefly recalled in the next section. Then, a new fundamental model will be proposed in order to overcome some of their restrictions and the first-order ($m = 1$) case will be examined.

1.2. CLASSICAL FUNDAMENTAL MODELS

The most common fundamental model of resonance is based on the pendulum approximation (Garfinkel, 1966; Jupp, 1982). Under an appropriate choice of units it can be expressed in terms of a momentum J and an angle ψ as a parameter independent Hamiltonian $\mathcal{K}^{(1)}$

$$\mathcal{K}^{(1)} = \tfrac{1}{2}J^2 - \cos m\psi. \tag{3}$$

Henrard and Lemaitre (1983) named $\mathcal{K}^{(1)}$ the first fundamental model and they pointed out its main drawback: most of the Hamiltonians in celestial mechanics are d'Alembertian series and this essential property is not reflected in $\mathcal{K}^{(1)}$. Their 'second fundamental model (SFM) of resonance' (Henrard and Lemaitre, 1983; Lemaitre, 1984) is much more suitable in this respect. Henrard and Lemaitre generalized a Hamiltonian originally proposed by Andoyer (1903) as a model of the 1:2 resonance in theory of asteroids and rediscovered in a more general context by Jefferys (1966). Starting from a three-parameter model

$$\overline{\mathcal{K}} = a_2 I^2 + a_1 I + b(2I)^{m/2}\cos m\phi, \tag{4}$$

they scaled variables I, ϕ and time in a way that finally reduced $\overline{\mathcal{K}}$ to a one-parameter SFM

$$\mathcal{K}^{(2)} = \tfrac{1}{2}J^2 + \alpha J + (2J)^{m/2}\cos m\psi. \tag{5}$$

It is worth noting that this reduction process requires that $a_2 \neq 0$ and $b \neq 0$. The former restriction excludes the so-called 'abnormal case' (Garfinkel, 1966). This remark is not a complaint, of course; it only recalls that the reduction of the number of parameters is always done at the expense of generality.

The SFM can be considered as a local, restricted case of the Colombo's top (CT) model (Henrard and Murigande, 1987); the phase space of SFM is a plane tangent to the sphere – the manifold of CT. This is yet another kind of a restriction. Only the fundamental models defined on a plane are discussed in this paper.

Although $\mathcal{K}^{(2)}$ has been successfully applied to various resonant problems, there exist remarkable phenomena it fails to reproduce. The most important of them is the occurrence of separatrix bifurcations also known as 'saddle connections' (Arnold et al., 1999) or 'heteroclinic bridges' (Ferraz-Mello et al., 1996). In order to extend the SFM, Shinkin (1995) proposed a Hamiltonian $\mathcal{K}^{(3)}$ that he named 'the third fundamental model':

$$\mathcal{K}^{(3)} = \frac{1}{2}J^2 + \alpha \prod_{k=1}^{4} (J + \beta_k)^{j_k/2} \cos m\psi, \tag{6}$$

where $j_1 + j_2 + j_3 + j_4 \leqslant m$. Although separatrix bifurcations may occur in the Shinkin's model, it still has some drawbacks like a high number of parameters β_k, j_k and the lack of the d'Alembertian character for $m > 1$.

2. Extended Fundamental Model (EFM)

Let us return to the Andoyer Hamiltonian (4). Instead of changing the amplitude of its periodic term, as did Shinkin, let us add a cubic term to the 'secular part':

$$\overline{\mathcal{M}} = a_3 I^3 + a_2 I^2 + a_1 I + b\,(2I)^{m/2} \cos m\phi. \tag{7}$$

This form may arise if one expands $\mathcal{H}_0$ in Taylor series retaining the third power of I, but one may also think about a possibly interesting alternative like the approximation of $\mathcal{H}_0$ by means of a cubic spline. The most important difference between the Hamiltonians (4) and (7) is due to the fact that for $b = 0$ and $I > 0$ a radial twist map generated by $\overline{\mathcal{K}}$ may have at most one critical circle with $\dot{\phi}(I) = 0$, whereas two such circles may occur in the mapping derived from $\overline{\mathcal{M}}$.

The Hamiltonian $\overline{\mathcal{M}}$ depends on four parameters, but their number can be reduced by two if a suitable choice of units is made. The reduced form of $\overline{\mathcal{M}}$, depending on two parameters, will be labeled $\mathcal{M}_m$ and named the EFM for the mth order resonance. In the following sections we will discuss the properties of the EFM $\mathcal{M}_1$.

2.1. REDUCTION OF PARAMETERS

For the first-order resonance we put $m = 1$ in the Hamiltonian (7), obtaining

$$\overline{\mathcal{M}}_1 = a_3 I^3 + a_2 I^2 + a_1 I + b\sqrt{2I} \cos \phi. \tag{8}$$

In order to reduce the number of parameters let us introduce new canonical variables J, ψ as well as a new time τ, such that

$$t = c_1 \tau, \qquad I = c_2 J, \qquad \phi = c_3 \psi + c_4 \pi. \tag{9}$$

The scaling constants c_k have to fulfill some *a priori* assumptions: in order to preserve the sign of the momentum, the direction of time flow, the 2π-periodicity of the Hamiltonian, and its cosine dependence on the angular variable, we assume

$$c_1 > 0, \qquad c_2 > 0, \qquad |c_3| = 1, \qquad c_4 \in \{0, 1\}. \tag{10}$$

The transformation $(I, \phi, t) \to (J, \psi, \tau)$ will be canonical (with a valence γ) if the classical condition

$$\gamma \left(I \, \mathrm{d}\phi - \overline{\mathcal{M}}_1 \, \mathrm{d}t \right) = J \, \mathrm{d}\psi - \mathcal{M}_1 \, \mathrm{d}\tau \tag{11}$$

is satisfied. Hence, substituting Equations (9), we obtain

$$\gamma c_2 c_3 \left(J \, \mathrm{d}\psi - \frac{c_1}{c_2 c_3} \overline{\mathcal{M}}_1 \, \mathrm{d}\tau \right) = J \, \mathrm{d}\psi - \mathcal{M}_1 \, \mathrm{d}\tau. \tag{12}$$

This implies $\gamma = 1/(c_2 c_3)$, and

$$\mathcal{M}_1 = \frac{c_1}{c_2 c_3} \overline{\mathcal{M}}_1. \tag{13}$$

Recalling the definitions (8) and (9) we may rewrite Equation (13) as follows:

$$\mathcal{M}_1 \; = \; a_3 \frac{c_1 c_2^2}{c_3} J^3 + a_2 \frac{c_1 c_2}{c_3} J^2 + a_1 \frac{c_1}{c_3} J + b \frac{c_1 \sqrt{2J}}{\sqrt{c_2}\, c_3} \cos\, (c_3 \psi + c_4 \pi). \tag{14}$$

In order to reduce the number of free parameters we need some assumptions about two of the coefficients a_k and b. Let us assume that $b \neq 0$ and $a_3 \neq 0$. The latter choice excludes a direct reduction of $\mathcal{M}_1$ to the SFM $\mathcal{K}^{(2)}$, but we gain the possibility of treating the abnormal case $a_2 = 0$.

Let us set

$$\frac{a_3 c_1 c_2^2}{c_3} = 1. \tag{15}$$

In virtue of the assumptions (10), this means

$$c_3 = \mathrm{sgn}(a_3), \qquad c_1 = \frac{1}{(|a_3| c_2^2)}, \tag{16}$$

and thus we can rewrite Equation (14) as

$$\mathcal{M}_1 \; = \; J^3 + \frac{a_2}{a_3 c_2} J^2 + \frac{a_1}{a_3 c_2^2 c_3} J + \frac{b \sqrt{2J}}{a_3 c_2^{5/2}} \cos\, \psi \, \cos\, c_4 \pi. \tag{17}$$

In order to reduce the last term of Equation (17), we set

$$c_2 = \left(\frac{b^2}{a_3^2} \right)^{1/5}, \tag{18}$$

and to account for the signs of b and a_3,

$$c_4 = \tfrac{1}{2}(1 + \mathrm{sgn}(a_3 b)). \tag{19}$$

All constants c_k are now defined, and there remain two parameters, say u and v, that depend on a_k and b. Thus we obtain the EFM

$$\mathcal{M}_1 = J^3 + \tfrac{1}{2}uJ^2 + vJ + \sqrt{2J}\cos\psi. \tag{20}$$

The two parameters of this model are

$$u = \frac{2\,a_2 c_1 c_2}{c_3} = \frac{2\,\mathrm{sgn}(a_3)\,a_2}{\left(b^2\,|a_3|^3\right)^{1/5}}, \tag{21}$$

$$v = \frac{a_1 c_1}{c_3} = \frac{\mathrm{sgn}(a_3)\,a_1}{\left(b^4\,|a_3|\right)^{1/5}}. \tag{22}$$

Now, we can proceed to the qualitative study of motion generated by $\mathcal{M}_1$.

2.2. CRITICAL POINTS

The variables (J, ψ) define a polar chart well known for its singularities at $J = 0$. As usual, we will pass to the canonical Cartesian variables (x, X):

$$x = \sqrt{2J}\sin\psi, \qquad X = \sqrt{2J}\cos\psi. \tag{23}$$

The Hamiltonian (20) expressed in terms of these variables

$$\mathcal{M}_1^* = \tfrac{1}{8}(x^2+X^2)^3 + \tfrac{1}{8}u(x^2+X^2)^2 + v(x^2+X^2) + X \tag{24}$$

is analytical at $x = X = 0$, and thus the system

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\partial \mathcal{M}_1^*}{\partial X}, \qquad \frac{\mathrm{d}X}{\mathrm{d}\tau} = -\frac{\partial \mathcal{M}_1^*}{\partial x} \tag{25}$$

is more convenient for a qualitative study.

Equations (25) admit critical points for $x = 0$, that is, at $\psi = 0$, or $\psi = \pi$. Then, the momentum X at the critical point is a real root of

$$\tfrac{3}{4}X^5 + \tfrac{1}{2}uX^3 + vX + 1 = 0. \tag{26}$$

According to classical theorems of algebra (Laurent, 1894) Equation (26) can only possess one, three, or five real roots, depending on the values of u and v. Evaluating the resultant of the polynomial from Equation (26) and of its first derivative

$$\tfrac{15}{4}X^4 + \tfrac{3}{2}uX^2 + v = 0, \tag{27}$$

we find a condition for the existence of double roots

$$\begin{aligned}&\tfrac{64}{84375}u^4v^3 - \tfrac{512}{28125}u^2v^4 + \tfrac{8}{3125}u^5 + \\ &\quad + \tfrac{1024}{9375}v^5 - \tfrac{8}{125}u^3v + \tfrac{32}{75}uv^2 + 1 = 0.\end{aligned} \tag{28}$$

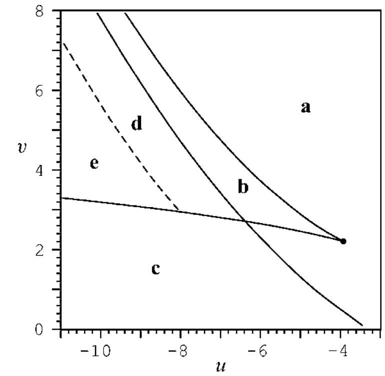

Figure 1. Bifurcation lines on the parametric plane (u, v).

The solution of Equation (28) is plotted with solid lines in Figure 1; it indicates the places on the parametric plane (u, v), where the saddle-node bifurcations occur.

Looking for a triple root of (26) we find

$$u = -\frac{5}{4^{1/5}}, \qquad v = \frac{15}{8}\,2^{1/5}, \qquad X = -\left(\frac{1}{2}\right)^{1/5}. \tag{29}$$

This point is marked as a black dot at the cusp in Figure 1.

We do not know the explicit algebraic solution of Equation (26), but it can be easily checked that the stable point S_0 (Figure 2) always exists, and if there are more of critical points, they satisfy

$$X_0 \leqslant X_1 \leqslant X_2 < 0 < X_3 \leqslant X_4, \tag{30}$$

$$0 < |X_2| \leqslant |X_1| < |X_3| \leqslant |X_4| \leqslant |X_0|, \tag{31}$$

where a critical point S_k has coordinates $x = 0$, $X = X_k$. All inequalities follow from the simple statement that the resultant of $f(X) = \frac{3}{4}X^5 + \frac{1}{2}uX^3 + vX + 1$ and $g(X) = f(-X)$ is $R(f, g) = \frac{243}{32} \neq 0$, and thus it is not possible to have a positive root X_n and a negative root X_m with the same absolute value. In this situation one

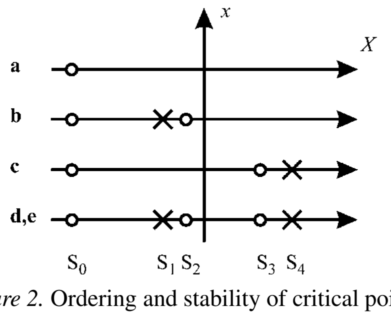

Figure 2. Ordering and stability of critical points.

may simply check the arrangement of five roots for any appropriate values of u, v and assert it as a generic property.

Similarly to the SFM (5) with $m = 1$ (Henrard and Lemaitre, 1983) the number of critical points of the phase flow increases only through the saddle-node bifurcations and thus each point's stability never changes. This statement can be verified through the inspection of the variational equations

$$\begin{pmatrix} \delta\dot{x}_k \\ \delta\dot{X}_k \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \mathcal{M}_1^*}{\partial x\,\partial X} & \frac{\partial^2 \mathcal{M}_1^*}{\partial X^2} \\ -\frac{\partial^2 \mathcal{M}_1^*}{\partial x^2} & -\frac{\partial^2 \mathcal{M}_1^*}{\partial x\,\partial X} \end{pmatrix}_{\mathrm{S}_k} \begin{pmatrix} \delta x_k \\ \delta X_k \end{pmatrix}, \tag{32}$$

evaluated at points S_k. Indeed, the eigenvalues of the system (32) become zero only at the points, where the condition (27) is satisfied.

Figure 2 lists all possible cases, with stable points marked as circles and unstable points as crosses. The arrangement of the critical points in Figure 2 reflects the properties (30) and (31). The labels from **a** to **e** refer to the respective regions in Figure 1. The same convention of labels, save for the case **f**, is maintained in Figure 3. The figure presents some examples of trajectories resulting from the equations of motion (25) for various values of parameters (u, v). Homoclinic and heteroclinic trajectories (separatrices) are marked with a thick line. Comparing the phase portraits **d** and **e**, one finds a qualitative difference that has no reference to the stability of critical points. Indeed, the number and the stability character of the points is similar in both cases; the difference comes from the arrangement of separatrices.

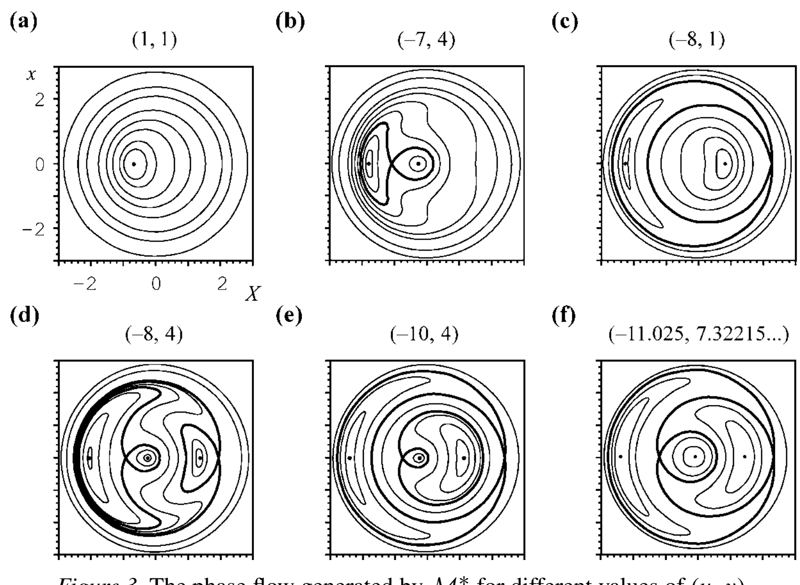

Figure 3. The phase flow generated by $\mathcal{M}_1^*$ for different values of (u, v).

The domain of Figure 1 has been restricted to $-11 < u < -3$ and the possibly interesting case of an abnormal resonance at $u = 0$ is not shown there. However, the remaining part of the plot results from a simple extrapolation of the curves. Thus, for $u = 0$ only the cases **a** or **c** are possible, with the bifurcation at $v \approx -1.557$.

2.3. SADDLE CONNECTIONS

The bifurcation case separating the cases (**d**) and (**e**) is labeled (**f**) in Figure 3. Using the nomenclature of Arnold et al. (1999), let us call it a saddle connections bifurcation. This type of bifurcation occurs when the values of the Hamiltonian $\mathcal{M}_1^*$ at both unstable points S_1 and S_4 are equal. Hence, in order to discuss the occurrence of saddle connections, one has to fix the value of one of the four quantities $\{u, v, X_1, X_4\}$, and then to solve the system of the following three equations:

$$\frac{1}{8}\left(X_1^6 - X_4^6\right) + \frac{1}{4}u\left(X_1^4 - X_4^4\right) + \frac{1}{2}v\left(X_1^2 - X_4^2\right) + X_1 - X_4 = 0, \tag{33}$$

$$\frac{3}{4}X_1^5 + \frac{1}{2}uX_1^3 + vX_1 + 1 = 0, \tag{34}$$

$$\frac{3}{4}X_4^5 + \frac{1}{2}uX_4^3 + vX_4 + 1 = 0. \tag{35}$$

As a measure of safety, one should check if a solution of this system obtained numerically does satisfy conditions $X_1 < 0$, $X_4 > 0$, and

$$15X_1^4 + 6uX_1^2 + 4v < 0, \tag{36}$$

$$15X_4^4 + 6uX_4^2 + 4v > 0, \tag{37}$$

that follow from the stability analysis; in other words, one should check whether both points are indeed unstable.

In order to derive the explicit relations between u or v and the values of X_1 and X_4, we can perform elementary, polynomial manipulations on Equations (33)–(35), arriving at

$$u = -2\left(X_1 + X_4\right)^2 + 3X_1X_4, \tag{38}$$

$$v = \frac{1}{4}\left[\left(X_1 + X_4\right)^4 - X_1X_4\left(\left(X_1 + X_4\right)^2 - 3X_1X_4\right)\right], \tag{39}$$

$$X_1X_4 = -4\left(X_1 + X_4\right)^{-3}. \tag{40}$$

The last equation of this system allows the reduction of the first two, and thus we are able to obtain the expression for u and v as functions of a single variable, say, $\sigma = X_1 + X_4$. The resulting system

$$2\sigma^5 + u\sigma^3 + 12 = 0, \tag{41}$$

$$\sigma^{10} - 4v\sigma^6 + 4\sigma^5 + 48 = 0, \tag{42}$$

can be treated with standard methods and the condition of vanishing resultant leads us to the final formula

$$\begin{aligned}&\tfrac{1}{7\,200\,000}u^{10} - \tfrac{17}{1\,800\,000}u^8 v + \tfrac{23}{90\,000}u^6 v^2 - \tfrac{43}{12\,500}u^4 v^3 + \\ &+ \tfrac{72}{3125}u^2 v^4 + \tfrac{151}{225\,000}u^5 - \tfrac{192}{3125}v^5 - \tfrac{31}{1500}u^3 v + \tfrac{4}{25}uv^2 + 1 = 0.\end{aligned} \tag{43}$$

The dashed line in Figure 1 represents the saddle connections bifurcation curve, plotted as the solution of Equation (43). It originates at

$$u = -\frac{10}{3^{1/5}}, \qquad v = \frac{55}{12 \times 3^{2/5}}, \tag{44}$$

and does not intersect the saddle-node bifurcation curves in any other point. Adding this line we establish the final partition of the parametric plane (u, v) into five qualitatively distinct regimes of motion.

3. Summary and Conclusions

The EFM for the first-order resonance has been proposed in Equation (20), or in the equivalent Equation (24). The model has been derived from a more general, m-order Hamiltonian (7) through the reduction of parameters. Qualitative changes in the phase flow of the extended model occur due to the saddle-node bifurcations and saddle connections for the values of parameters given by Equations (28) and (43), respectively.

Is there any interest in using the EFM rather than the SFM of Henrard and Lemaitre (1983)? A definite answer may only come from the actual application of EFM in the studies of relevant resonant problems, most probably in the secular resonances of the asteroid theory. Ferraz-Mello et al. (1996) published the Poincaré maps for the 3/1 and 2/1 mean motion resonances combined with the ν_5 secular resonance in the Sun–Jupiter–asteroid problem. All of their 3/1 figures and one of the 2/1 plots do resemble a perturbed model shown in **d** and **e** of Figure 1 of this paper. The authors emphasize the importance of the saddle connections in the migration of asteroids between three libration regimes of their eccentricities. Similar patterns can be found in the paper of Moons and Morbidelli (1995). Both papers avoid the use of the classical Laplace–Leverrier expansion thanks to different semi-analytical tools. On the other hand, the results obtained by authors who rely on the fourth degree Laplace–Leverrier expansion of a perturbing function, like Wisdom (1982) or Yoshikawa (1987), are always SFM-similar.

The author's first attempts to derive the **d** or **e** cases of the EFM directly from the Leverrier (1855) tables have failed. Apparently, the appearance of the fully developed EFM is related to the values of eccentricity that lie outside the radius of convergence for the Laplace–Leverrier expansion. In this situation, the task of providing an illustration for the model just proposed amounts to a problem that merits a separate study. The author is grateful to an anonymous reviewer for pointing out this difficulty.

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