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ORIGINAL ARTICLE

Critical inclination in the main problem of a massive satellite

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Abstract The classical problem of the critical inclination in artificial satellite theory has been extended to the case when a satellite may have an arbitrary, significant mass and the rotation momentum vector is tilted with respect to the symmetry axis of the planet. If the planet's potential is restricted to the second zonal harmonic, according to the assumptions of the main problem of the satellite theory, two various phenomena can be observed: a critical inclination that asymptotically tends to the well known negligible mass limit, and a critical tilt that can be attributed to the effect of transforming the gravity field harmonics to a different reference frame. Stability of this particular solution of the two rigid bodies problem is studied analytically using a simple pendulum approximation.

Keywords Analytical methods · Critical inclination · Rigid body rotation

1 Introduction

The phenomenon of the critical inclination in artificial satellite theory has been known since 1950's. An in-depth overview and an extensive list of related papers can be found in Coffey et al. (1986, 1994). According to the first order theory, the secular perturbations in the argument of pericentre g vanish if the orbit's inclination I is the root of

$$1 - 5\cos^2 I = 0. \tag{1}$$

In the second order approximation, the phenomenon clearly becomes a resonance; with $J_2 > 0$ and all other harmonics neglected two kinds of frozen orbits appear: stable ones with $g \in \{90°, 270°\}$ and unstable with $g \in \{0°, 180°\}$.

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The critical inclination was discovered and studied under the assumptions that the satellite's mass M_0 is negligible with respect to the planet's mass M_p. After a minor modification of the gravity parameter, the theory remains valid even for a significant mass of the satellite; this requires, however, that the torques, affecting the spin axes of both bodies, are neglected. It is not unreasonable to expect that the critical inclination in artificial satellite theory is a limiting case of some more general phenomenon occurring in the problem of two rigid bodies. Thus, in the present paper we consider the extended main problem of the artificial satellite where two restrictions have been suppressed: a spherical satellite may have an arbitrary mass (even much larger than the oblate "planet") and the "planet" may rotate around the axis that is tilted with respect to its symmetry axis. In order to avoid any suggestions concerning the mass ratio of the two bodies, we will use the term "central body" or "primary" instead of the "planet". Throughout the paper we use the formalism and, to large extent, the notation adopted from the fundamental paper of Kinoshita (1972).

Using the system of a nonspherical body and a homogenous sphere as a first approximation to the full problem of two rigid bodies is quite common in the binary asteroids modeling. On can find it in papers by Scheeres (2001, 2004a), Breiter et al. (2005), and—partially—in (Scheeres 2002, 2004b; Koon et al. 2004).

2 Reference frames and variables

Let us introduce two basic reference frames, both having their origins at the centre of mass of an axially symmetric central body O: the *body frame* $Oxyz$, with the basis unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, and the *fixed frame* $OXYZ$ with the $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$, $\hat{\mathbf{Z}}$ unit vectors. Let $\hat{\mathbf{z}}$ be directed along the shortest axis of the central body, $\hat{\mathbf{x}}$ towards an arbitrary point on the equator, and $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$. The total momentum of the system $\mathbf{\Gamma}$ (the sum of the central body angular momentum $\mathbf{G}_1$ and of the orbital angular momentum of the sphere $\mathbf{G}$) is constant, i.e.

$$\mathbf{\Gamma} = \mathbf{G}_1 + \mathbf{G} = \mathrm{I}_\mathrm{p}\,\boldsymbol{\omega} + \mathbf{R} \times \mathbf{P} = \mathbf{const}, \tag{2}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rotating central body, $\mathbf{R}$ and $\mathbf{P}$ are the relative position of the sphere and its canonically conjugate momentum respectively, and I_p is the matrix of inertia

$$I_\mathrm{p} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix}. \tag{3}$$

Following Kinoshita (1972) we choose the $\hat{\mathbf{Z}}$ vector of the fixed frame directed along $\mathbf{\Gamma}$, the $\hat{\mathbf{X}}$ vector—orthogonal to $\hat{\mathbf{Z}}$—directed to some arbitrary fixed point, and $\hat{\mathbf{Y}} = \hat{\mathbf{Z}} \times \hat{\mathbf{X}}$.

Two sets of canonical variables are required in our problem: one for the rotation of the central body and one for the orbital motion of the sphere with respect to the central body. The rotation will be described by means of the Serret–Andoyer variables (Deprit and Elipe 1993) with the momenta
G_1—the magnitude of the rotation angular momentum,
L_1—the projection of $\mathbf{G}_1$ on the polar axis Oz

$$L_1 = \mathbf{G}_1 \cdot \hat{\mathbf{Z}} = G_1 \cos J_1, \tag{4}$$

H_1—the projection of $\mathbf{G}_1$ on the fixed axis OZ

$$H_1 = \mathbf{G}_1 \cdot \hat{\mathbf{Z}} = G_1 \cos I_1, \tag{5}$$

and their conjugate angles g_1, ℓ_1, h_1. The geometrical meaning of the angles can be easily deduced from the rotations sequence required to transform any position vector, say $\mathbf{R}$, with the components expressed in the fixed frame, to the body frame; the resulting vector $\mathbf{r}$ will have the components

$$\mathbf{r} = R_3(\ell_1)\, R_1(J_1)\, R_3(g_1) R_1(I_1)\, R_3(h_1)\, \mathbf{R}, \tag{6}$$

where R_1 and R_3 are the usual matrices of rotation

$$R_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}, \quad R_3(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

An appropriate figure can be found in Kinoshita (1972, Figs. 1 and 5).

Orbital motion can be described in terms of the Delaunay variables, consisting of the mean anomaly ℓ, argument of pericentre g, longitude of the ascending node h (all measured in the fixed frame), and of their conjugate momenta

$$L = m\sqrt{\mu\, a} = m\, n\, a^2, \tag{8}$$

$$G = ||\mathbf{G}|| = L\,\eta = m\, n\, a^2 \sqrt{1-e^2}, \tag{9}$$

$$H = \mathbf{G} \cdot \hat{\mathbf{Z}} = G\, \cos I, \tag{10}$$

where

$$m = \frac{M_{\mathrm{p}}\, M_0}{M_{\mathrm{p}} + M_0}, \quad \mu = k^2\,(M_{\mathrm{p}} + M_0), \tag{11}$$

M_0 being the mass of the orbiting sphere, k—the Gaussian gravity constant, a—orbital major semi-axis, e—orbital eccentricity, and n is the mean motion. Using the true anomaly f, we can express the position vector $\mathbf{R}$ of the sphere as

$$\mathbf{R} = R_3(-h)\, \mathrm{R}_1(-I)\, R_3(-f-g)\ (r, 0, 0)^{\mathrm{T}}. \tag{12}$$

Equations (6) and (12) allow to express the position of the sphere with respect to the body frame $\mathbf{r}$ as a function of the Delaunay and Serret–Andoyer variables.

According to the well known properties of the system (Kinoshita 1972), the choice of $\hat{\mathbf{Z}} = \hat{\mathbf{\Gamma}}$ implies important consequences:

1. Orbital plane and the plane normal to $\mathbf{G}_1$ intersect along the common line of the nodes on the *invariable plane* OXY, and

$$h_1 - h = \pi. \tag{13}$$

2. Momenta G_1, H_1, G, H are not independent, because of

$$G^2 - H^2 = G_1^2 - H_1^2, \quad H + H_1 = \alpha = \text{const.} \tag{14}$$

or, equivalently,

$$G \sin I - G_1 \sin I_1 = 0, \quad G \cos I + G_1 \cos I_1 = \alpha = \text{const.} \tag{15}$$

This means that given the value of α, the motion of L, G, L_1, G_1 and of their conjugate angles can be studied separately from H, h, H_1, and h_1; the number of degrees of freedom is effectively reduced from 6 to 4.

3 Hamiltonian and equations of motion

The Hamiltonian function of the system is

$$\mathcal{H} = \frac{G_1^2 - L_1^2}{2\,A} + \frac{L_1^2}{2\,C} - \frac{m^3\,\mu^2}{2\,L^2} + J_2\,\frac{m\,\mu\,a_{\mathrm{p}}^2}{r^3}\,P_2(\sigma), \tag{16}$$

where a_{p} is the equatorial radius of the primary, P_j is the Legendre polynomial of degree j, and $\sigma = \hat{\mathbf{r}} \cdot \hat{\mathbf{z}}$ is the sine of latitude of the sphere in the body frame.

The moments of inertia are related to the second zonal harmonic coefficient

$$C - A = J_2 M_{\mathrm{p}} a_{\mathrm{p}}^2. \tag{17}$$

Substituting this relation and expanding the Hamiltonian in power series of J_2, we may partition $\mathcal{H}$ into

$$\mathcal{H} = \mathcal{H}_0 + J_2\,\mathcal{H}_1 + \tfrac{1}{2}J_2^2\,\mathcal{H}_2 + O\left(J_2^3\right), \tag{18}$$

$$\mathcal{H}_0 = \frac{G_1^2}{2\,C} - \frac{m^3\,\mu^2}{2\,L^2}, \tag{19}$$

$$\mathcal{H}_1 = \frac{M\,a_{\mathrm{p}}^2\,(G_1^2 - L_1^2)}{2\,C^2} + \frac{m\,\mu a_{\mathrm{p}}^2}{r^3}\,P_2(\sigma), \tag{20}$$

$$\mathcal{H}_2 = \frac{M^2\,a_{\mathrm{p}}^4\,(G_1^2 - L_1^2)}{C^3}. \tag{21}$$

The unperturbed Hamiltonian $\mathcal{H}_0$ describes the system physically equivalent to the problem of two spheres: all momenta are integrals of motion and all angles are constant, save for

$$\dot{\ell} = \frac{\partial \mathcal{H}_0}{\partial L} = \frac{m^3\,\mu^2}{L^3} \equiv n, \tag{22}$$

$$\dot{g}_1 = \frac{\partial \mathcal{H}_0}{\partial G_1} = \frac{G_1}{C} \equiv n_1. \tag{23}$$

These two frequencies are present in the definition of the Lie derivative (Deprit 1969)

$$L_0 F = (F; \mathcal{H}_0) = n\,\frac{\partial F}{\partial l} + n_1\,\frac{\partial F}{\partial g_1}, \tag{24}$$

where F is any function and $(;)$ stands for the canonical Poisson bracket.

4 Rigid body potential in the inertial frame

Before we proceed to the perturbation treatment, let us look at the spheroid from another perspective. The usual treatment of the orbital motion amounts to transforming the position of the sphere from the inertial frame XYZ to the central body frame xyz. As the alternative, we propose to do the opposite, transforming the central body potential to the inertial frame XYZ. However, making use of the property (13), we transform the potential to the *nodal reference frame* $X^* Y^* z$ with the axis OX^* directed towards the ascending node of the sphere's orbit.

Using theorems about the transformation of Legendre functions under rotation we obtain

$$P_2(\sigma) = C^*_{2,0} P_2(\sigma^*) + \sum_{j=1}^{2} P_{2,j}(\sigma^*) \left(C^*_{2,j} \cos j\lambda^* + S^*_{2,j} \sin j\lambda^* \right), \tag{25}$$

where $\sigma^* = \hat{\mathbf{R}} \cdot \hat{\mathbf{Z}}$ is the sine of latitude with respect to the invariant plane and λ^* is the longitude measured from the orbital ascending node. Introducing symbols $s_{I1} = \sin I_1$, $c_{I1} = \cos I_1$ and $c_1 = \cos J_1$, we can express the transformed potential coefficients as

$$\begin{aligned} C^*_{2,0} &= P_2(c_{I1}) P_2(c_1) - \frac{1}{3} P_{2,1}(c_{I1}) P_{2,1}(c_1) \cos g_1 \\ &\quad + \frac{1}{12} P_{2,2}(c_{I1}) P_{2,2}(c_1) \cos 2g_1, \end{aligned} \tag{26}$$

$$C^*_{2,1} = -\frac{c_{I1}}{3} P_{2,1}(c_1) \sin g_1 + \frac{s_{I1}}{6} P_{2,2}(c_1) \sin 2g_1, \tag{27}$$

$$\begin{aligned} C^*_{2,2} &= -\frac{1}{24} P_{2,2}(c_{I1}) P_2(c_1) - \frac{1}{18} P_{2,1}(c_{I1}) P_{2,1}(c_1) \cos g_1 \\ &\quad - \frac{1 + c^2_{I1}}{24} P_{2,2}(c_1) \cos 2g_1, \end{aligned} \tag{28}$$

$$\begin{aligned} S^*_{2,1} &= \frac{1}{3} P_{2,1}(c_{I1}) P_2(c_1) - \frac{1 - 2c^2_{I1}}{3} P_{2,1}(c_1) \cos g_1 \\ &\quad - \frac{1}{18} P_{2,1}(c_{I1}) P_{2,2}(c_1) \cos 2g_1, \end{aligned} \tag{29}$$

$$S^*_{2,2} = -\frac{s_{I1}}{6} P_{2,1}(c_1) \sin g_1 - \frac{c_{I1}}{12} P_{2,2}(c_1) \sin 2g_1. \tag{30}$$

Thus the second zonal harmonic of the potential in the $X^* Y^* Z$ frame becomes

$$V^*_2 = \frac{m\,\mu\,a^2_p}{r^3} \sum_{j=0}^{2} P_{2,j}(\sigma^*) \left(J_2 C^*_{2,j} \cos j\lambda^* + J_2 S^*_{2,j} \sin j\lambda^* \right). \tag{31}$$

Assuming the unperturbed rotation with constant I_1, J_1 and n_1, we can see the rigid spheroid in the new reference frame as an object continuously changing its shape due to the time-dependence of $C^*_{2,j}$ and $S^*_{2,j}$ coefficients.

Looking forward to the application of averaging technique, let us ask about an average potential of the transformed body. Rejecting all terms that are periodic functions of g_1 we obtain

$$\bar{C}^*_{2,0} = P_2(c_{I1}) P_2(c_1) = \tfrac{1}{4} \left(2 - 3s^2_{I1} \right) \left(2 - 3s^2_1 \right), \tag{32}$$

$$\bar{C}^*_{2,2} = -\frac{1}{24} P_{2,2}(c_{I1}) P_2(c_1) = -\frac{s^2_{I1}}{8} \left(2 - 3s^2_1 \right), \tag{33}$$

$$\bar{S}^*_{2,1} = \frac{1}{3} P_{2,1}(c_{I1}) P_2(c_1) = \frac{s_{I1} c_{I1}}{2} \left(2 - 3s^2_1 \right), \tag{34}$$

as the only nonvanishing terms. As it should be expected, if the rotation state is the shortest axis mode and the angular momentum $\boldsymbol{G}_1$ is normal to the invariant plane ($s_1 = \sin J_1 = 0$ and $s_{I1} = 0$), we recover the original potential with $C^*_{2,0} = \bar{C}^*_{2,0} = 1$

and all remaining coefficients equal to zero. But even more noteworthy is the fact that all the averaged coefficients have a common factor $P_2(c_1)$. Thus, if

$$\sin J_1 = \sqrt{\frac{3}{2}}, \tag{35}$$

the averaged potential vanishes. In further discussion we will call *the critical tilt* each of the two values $J_1 \approx 54^\circ\!.74$ and $J_1 \approx 125^\circ\!.26$, for which the potential of a rotating spheroid in the nodal frame differs from the point mass potential only by purely periodic terms in the quadrupole approximation.

5 First order normalization

Using the classical Lie transformation method to normalize the Hamiltonian function of our problem (Deprit, 1969), we will obtain the first order normalized Hamiltonian $\mathcal{K}_1$ as the sum of these terms of $\mathcal{H}_1$ that belong to the kernel of L_0 and so do not depend on the mean variables ℓ and g_1, i.e.

$$L_0 \mathcal{K}_1 = 0, \tag{36}$$

and $\mathcal{H}_1 - \mathcal{K}_1$ is purely periodic. This approach, however, requires two important assumptions:

1. We exclude all resonance that might occur due to the commensurability of n and n_1.
2. We assume the fast rotation with sufficiently large ratio G_1/G, because otherwise we obtain yet another kind of resonance leading to abnormally large amplitudes of g_1-dependent terms in the generator $\mathcal{W}_1$ defined through

$$L_0 \mathcal{W}_1 = \mathcal{H}_1 - \mathcal{K}_1. \tag{37}$$

If none of these restrictions is violated, we obtain the new Hamiltonian $\mathcal{K}_1$ as

$$\begin{aligned} \mathcal{K}_1 \ &= \frac{1}{2} M_{\mathrm{p}} a_{\mathrm{p}}^2 n_1^2 s_1^2 + \frac{m\,\mu\, a_{\mathrm{p}}^2}{a^3} \left[\bar{C}^*_{2,0} \left\langle \frac{a^3\, P_2(\sigma^*)}{r^3} \right\rangle \right. \\ &\left. + \bar{S}^*_{2,1} \left\langle \frac{a^3\, P_{2,1}(\sigma^*)}{r^3} \, \sin\lambda^* \right\rangle + \bar{C}^*_{2,2} \left\langle \frac{a^3\, P_{2,2}(\sigma^*)}{r^3} \, \cos 2\lambda^* \right\rangle \right]. \end{aligned} \tag{38}$$

The average values in Eq. (38) are obtained by elementary quadratures, taking into account

$$\langle F(f) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{F\, r^2}{a^2 \eta}\, df, \tag{39}$$

and recalling that in the nodal frame

$$\begin{pmatrix} \sqrt{1-(\sigma^*)^2}\, \cos\lambda^* \\ \sqrt{1-(\sigma^*)^2}\, \sin\lambda^* \\ \sigma^* \end{pmatrix} = R_1(-I)\, R_3(-f-g) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{40}$$

So we obtain

$$\left\langle \frac{a^3\, P_2(\sigma^*)}{r^3} \right\rangle = -\frac{2-3\,s^2}{4\,\eta^3}, \tag{41}$$

well known from the classical satellite theory, and

$$\left\langle \frac{a^3\, P_{2,1}(\sigma^*)}{r^3}\ \sin\lambda^* \right\rangle = \frac{3\, s\, c}{2\, \eta^3}, \tag{42}$$

$$\left\langle \frac{a^3\, P_{2,2}(\sigma^*)}{r^3}\ \cos 2\lambda^* \right\rangle = \frac{3\, s^2}{2\, \eta^3}, \tag{43}$$

where $s = \sin I$ and $c = \cos I$.

Collecting all results, we have the new Hamiltonian

$$\mathcal{K} = \mathcal{K}_0 + J_2 \mathcal{K}_1 + O\left(J_2^2\right), \tag{44}$$

$$\mathcal{K}_0 = \frac{G_1^2}{2\, C} - \frac{m^3 \mu^2}{2\, L^2}, \tag{45}$$

$$\mathcal{K}_1 = \frac{M_{\mathrm{p}}\, a_{\mathrm{p}}^2\, G_1^2\, s_1^2}{C^2} + \frac{a_{\mathrm{p}}^2 m \mu}{4 a^3 \eta^3} \left(-\bar{C}_{2,0}^* (2 - 3 s^2) + 6 s \left(\bar{S}_{2,1}^*\, c + \bar{C}_{2,2}^*\, s \right) \right). \tag{46}$$

Introducing the mutual inclination J as an angle between $\mathbf{G}$ and $\mathbf{G}_1$

$$J = I_1 + I, \tag{47}$$

we can simplify the final form of $\mathcal{K}_1$ that becomes

$$\mathcal{K}_1 = \frac{M\, a_{\mathrm{p}}^2\, G_1^2\, s_1^2}{C^2} - \frac{a_p^2 m \mu}{2 a^3 \eta^3}\, P_2(c_1) P_2(\cos J). \tag{48}$$

In the limiting case of the shortest axis rotation and negligible mass M_0, when $m \to M_0$, $J = I$, and $J_1 = 0$, the second term of (48) becomes the classical satellite theory Hamiltonian (Brouwer 1959). The Hamiltonian $\mathcal{K}_1$ depends only on Andoyer and Delaunay momenta that are prime integrals of the normalized system. It means that all Andoyer and Delaunay angles are linear functions of time.

6 Critical inclination and critical tilt

The frequency of the mean argument of pericentre in the inertial reference frame can be obtained from $\mathcal{K}_1$

$$\dot{g} = J_2\, \frac{\partial \mathcal{K}_1}{\partial G} = J_2\, \frac{3 n\, a_{\mathrm{p}}^2}{8\, a^2\, (1 - e^2)}\, P_2(c_1)\, \Psi(I, J), \tag{49}$$

where

$$\Psi(I, J) = 1 + 3\ \cos 2J + 2 c\, s^{-1} \sin 2J. \tag{50}$$

There are two possibilities that lead to the situation when $\dot{g} = 0$. The first one is $P_2(c_1) = 0$; this condition is satisfied at the critical tilt defined in Sect. 4. Its meaning is clearly understandable: secular perturbations in the argument of pericentre vanish because the potential differs from the Keplerian case only by short-periodic terms. Recalling that $\cos J_1 = L_1/G_1$ we observe, that $L_1 = \text{const}$ not only as a mean variable, but it is also the prime integral of the original system due to the absence of ℓ_1 in the Hamiltonian $\mathcal{H}$, as a consequence of the axial symmetry of the central body.

The second condition for $\dot{g} = 0$—entirely independent on the first one—is obviously $\Psi = 0$. As we will see, it generalizes the classical critical inclination condition. In purely geometrical terms, the critical inclination I is a function of the mutual inclination J (Fig. 1a), or of the obliquity I_1 (Fig. 1b). The relation between the obliquity and mutual inclination is shown in Fig. 1c. The classical limit is recovered when the obliquity $I_1 = 0$, i.e. when $J = I$ (the dashed line in Fig. 1a). In this case $\Psi = 2\,(1-5\,c^2)$ and the critical inclination values are $I \approx 63^\circ\!.43$ and $I \approx 116^\circ\!.37$.

One should remember, however, that the geometry of I, I_1 and J is a function of momenta G and G_1. If we substitute Eq. (15) into the expression of Ψ, the critical inclination becomes a function of one physical parameter: the ratio G/G_1. This dependence is shown in Fig. 1d.

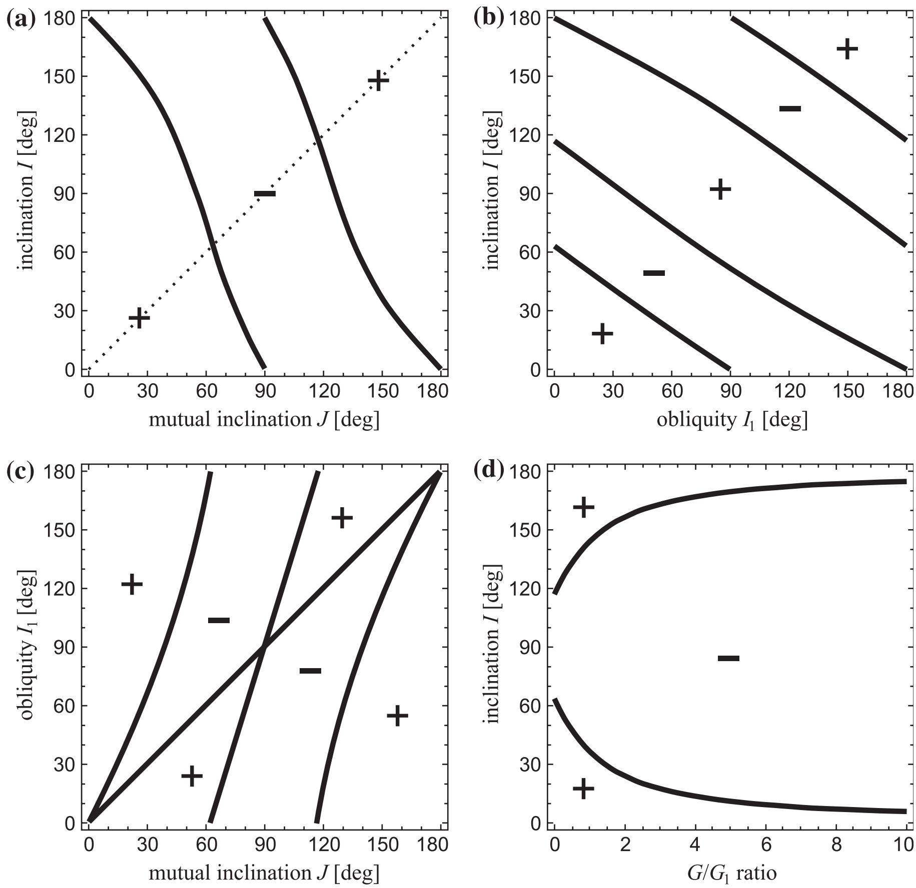

Fig. 1 Critical inclination in the first-order approximation. Plus and minus signs indicate the sign of secular perturbations in g when $P_2(c_1) > 0$

7 Stability of critical inclination

Within the first order approximation presented in Sect. 6, the critical inclination on a (g, G) phase plane is a line of critical points $G = \text{const}$. Similarly to the classical, negligible mass satellite problem, it is only at the second order of perturbation theory that we start observing isolated critical points with complementary stability indices, i.e. we pass from the critical inclination phenomenon to the notion of frozen orbits. But in our present generalization, reaching the second order becomes a real challenge: it is not feasible in a closed form, the perturbing potential is much more complicated, and the number of parameters is higher than in the small satellite case. For these reasons we present very limited results concerning the stability of the frozen orbits, obtained within a crude approximation. The second order normalization was performed according to the Lie–Deprit method (Deprit, 1969), with

$$\mathcal{K}_2 = \langle \mathcal{H}_2 + (\mathcal{H}_1 + \mathcal{K}_1; \mathcal{W}_1) \rangle , \tag{51}$$

but all short-periodic terms were a priori rejected before the evaluation of the Poisson bracket and ultimately we reduced the Hamiltonian

$$\mathcal{K} = \mathcal{K}_0 + J_2 \mathcal{K}_1 + \frac{J_2^2}{2} \mathcal{K}_2 \tag{52}$$

to a standard pendulum model

$$\mathcal{K}_\mathrm{p} = \frac{\mathcal{A}}{2} \hat{G}^2 + \mathcal{B} \cos 2g, \tag{53}$$

where

1. $\mathcal{A}$ is $J_2\, \partial^2 \mathcal{K}_1 / \partial G^2$ evaluated at the values of G/G_1 and I that satisfy $\Psi = 0$,
2. $\mathcal{B}$ is the coefficient of the $\cos 2g$ term in the $\frac{1}{2} J_2^2 \mathcal{K}_2$, evaluated at the values of G/G_1 and I that satisfy $\Psi = 0$, and truncated at the second power of the orbital eccentricity,
3. $\hat{G} = G - G_\mathrm{cr}$, where G_cr is the value of orbital momentum that satisfies $\Psi = 0$.

Even with so strong approximations, the pendulum Hamiltonian $\mathcal{K}_\mathrm{p}$ contains hundreds of terms and is too long to be quoted in this paper. However, one should be aware that some terms of $\mathcal{B}$ contain possibly resonant denominators $n \pm n_1$, $n \pm 2n_1$, and $2n \pm n_1$. Both n and n_1 are positive by definition, hence in the subsequent discussion the resonances $n \approx n_1$, $n \approx 2n_1$, and $2n \approx n_1$ have to excluded.

The approximate Hamiltonian $\mathcal{K}_\mathrm{p}$ leads to the equations of motion with two pairs of equilibria: $\mathrm{E1} = (g \in \{0, 180°\}, \hat{G} = 0)$, and $\mathrm{E2} = (g \in \{90°, 270°\}, \hat{G} = 0)$ with a complementary stability. The equilibrium E1 is stable (and E2—unstable) if $\mathcal{B}/\mathcal{A} < 0$. It can be easily verified that

$$\operatorname{sgn} \mathcal{A} = \operatorname{sgn} P_2(c_1), \tag{54}$$

hence the stability problem is reduced to the study of the sign of $\mathcal{B}$. On the other hand, according to Eq. (54) we will have to avoid the values of J_1 that are too close to the critical tilt $P_2(c_1) = 0$. The sign of $\mathcal{B}$ is determined by two parameters: the ratios G/G_1 and n/n_1.

The results are presented in Figs. 2 and 3. Thick dashed lines in these figures indicate the resonances between n and n_1 where the present analysis is not valid. Fig. 2 presents the case of an "average oblate" body when $P_2(c_1) > 0$. In Figure 2a one sees the case with no tilt ($J_1 = 0$). When $G/G_1 = 0$, we recover the classical satellite case of stable E2 orbits regardless of the n/n_1 ratio. But when the orbital momentum is not negligible, some these orbits may become unstable. However, when the tilt is significant (Fig. 2b) even the $G/G_1 = 0$ may lead to the occurrence of unstable E2 orbits; the situation is actually equivalent to the problem of critical inclination (or frozen orbits) around a significantly three-axial oblate body (i.e. one with J_2 comparable to the $J_{2,2}$ coefficient, but both considered small). Figure 3 presents two cases with $P_2(c_1) < 0$, ie. for an "average prolate" body. We have intentionally avoided the values of J_1 close to the critical tilt, where the pendulum approximation breaks down because of the vanishing $\mathcal{A}$.

As for the critical tilt phenomenon, it is not of the resonance type, because there are no isolated unstable critical points on the (ℓ_1, L_1) plane.

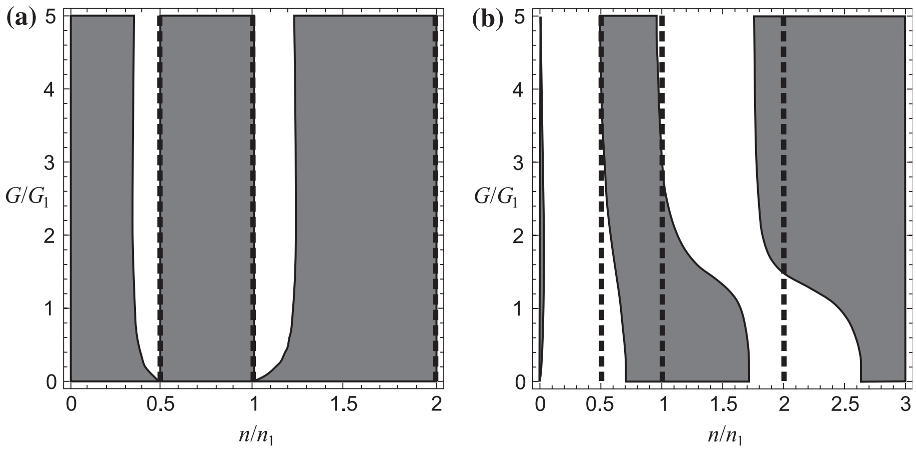

Fig. 2 Stability of frozen orbits when $P_2(c_1) > 0$, for two sample values of tilt: $J_1 = 0$ (**a**) and $J_1 = 45°$ (**b**) E2 equilibria are stable inside the grey zones

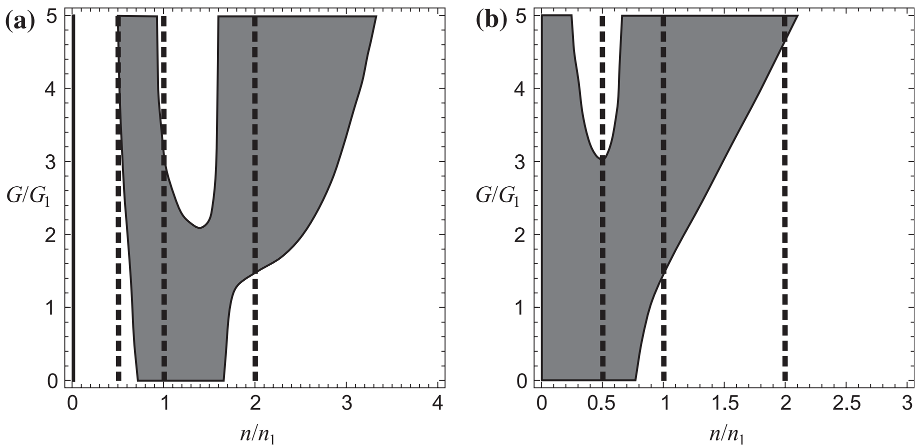

Fig. 3 Stability of frozen orbits when $P_2(c_1) < 0$, for two sample values of tilt: $J_1 = 60°$ (**a**) and $J_1 = 90°$ (**b**) E1 equilibria are stable inside the grey zones

8 Conclusions

The problem of frozen line of the apsides in the spheroid and sphere case becomes much more profound than its artificial satellite counterpart if we suppress at least one of the usual approximations: short axis rotation and negligible mass of the satellite. Without the short axis rotation assumption, the problem becomes actually equivalent to the motion around a three-axial body ("tesseral resonances" appear) and is accompanied by the phenomenon of critical tilt. On the other hand, increasing the satellite's mass we shift the critical inclination value towards 0 (prograde motion) or 180° (retrograde motion). Further studies concerning the stability of frozen orbits are still required, because the pendulum approximation and $O(e^2)$ truncation that were used in this paper should rather be treated as nothing more than a "quick look" model. Obviously, introducing more harmonics in the potential of the central body may change the stability of frozen orbits as it happens in the artificial satellite theory (Coffey et al. 1994). Nevertheless, the growing number of known binary asteroids with comparable masses of their components indicates that the problems discussed in this paper are more than "academic problems".

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